Tutorial 27 - Finding MIMO

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When multiple input/multiple output (MIMO) systems were described in the mid-to-late 1990s by Gerard Foschini and others, [1] the astonishing bandwidth efficiency of such techniques seemed to be in violation of the Shannon limit. But, there is no violation because the diversity and signal processing employed with MIMO transforms a point-to-point single channel into multiple parallel or matrix channels, hence in effect multiplying the capacity. MIMO offers higher data rates as well as spectral efficiency. So clear is this advantage that many standards have already incorporated MIMO. ITU uses MIMO in the High Speed Downlink Packet Access (HSPDA), part of the UMTS standard. MIMO is also part of the 802.11n standard used by your wireless router as well as 802.16 for Mobile WiMax used by your cell phone. The LTE standard also incorporates MIMO.

What is MIMO as compared to a traditional communications channel? A traditional communications link, which we call a single-in-single-out (SISO) channel, has one transmitter and one receiver. But instead of a single transmitter and a single receiver we can use several of each. The SISO channel then becomes a multiple-in-multiple-out, or a MIMO channel; i.e. a channel that has multiple transmitters and multiple receivers.

What does MIMO offer over a traditional SISO channel? To examine this question, we will first look at the capacity of a SISO link, which is specified in the number of bits that can be transmitted over it as measured by the very important metric, (b/s/Hz).

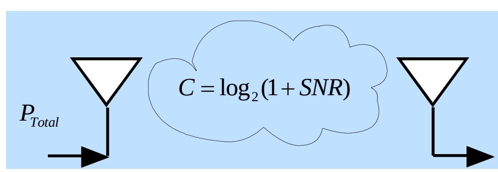


Figure 27.1 – Claude Shanon's SISO channel capacity

The capacity of a SISO link is a function simply of the channel SNR as given by the Equation in Figure 27-1. This capacity relationship was of course established by Claude Shannon [2] and is also called the information-theoretic capacity. The SNR in this equation is defined as the total power divided by the noise power.

Example 1: What is the capacity of a channel with an SNR of 10 dB.

$$C = log2\left(1+10\right) = 3.46\frac{b}{s}/Hz \qquad 27.1$$

This relationship says that an increase of power by a factor of 10 times, i.e. a SNR of 20 dB will increase the capacity to 6.65 b/s/Hz, a less than doubling of capacity. A one-hundred-time increase in power will increase the channel capacity to only 9.96 b/s/Hz, approximately a tripling of capacity. The capacity is increasing as a log function of the SNR, which is a slow increase. Clearly increasing the capacity by any significant factor takes an enormous amount of power in a SISO channel. Wouldn't it be nice if we can increase the capacity instead by a linear function of power; 10 times increase in power, 10 times increase in capacity! Perhaps we can do this with MIMO.

With MIMO, we move to a different paradigm of channel capacity. To give you a feel for what is possible, if we add six antennas on both transmit and receive side, we can achieve the same capacity as using 100 times more power than in the SISO case. So what did we do here? We just made the transmitter and receiver more complex, with no increase in power at all. We got the same performance as increasing the power 100 times. Quite amazing, and worth examining closely.

In Figure 27.2, we see the comparison of SISO and MIMO systems using the same power. MIMO capacity increases linearly with the number of antennas, where SISO/SIMO/MISO systems all increase only logarithmically.

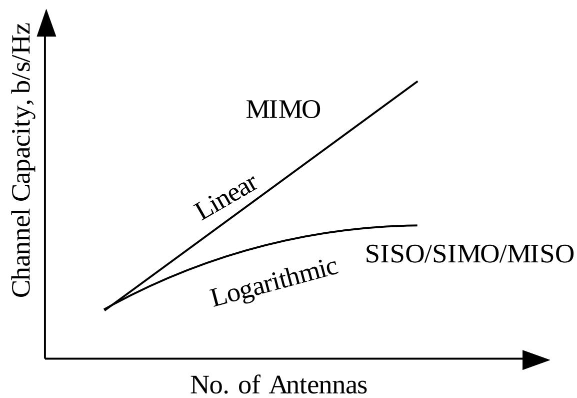


Figure 27.2 – MIMO offers a way to increase capacity without increasing power.

At conceptual level, you can think of MIMO as a way of enhancing the dimensions of communication. However, MIMO is not Multiple Access. It is not like FDMA because all “channels” use the same frequency, and it is not TDMA because all channels operate simultaneously. There is no way to separate the channels in MIMO by code, as is done in CDMA and there are no steerable beams or smart antennas as in SDMA. MIMO exploits an entirely different dimension.

What we have here is not one channel but multiple, $N_R \times N_T$, if N_T is the number of antennas on the transmit side and N_R, on the receive side. Somewhat like the idea of OFDM, the signal travels over multiple paths and then is recombined in a smart way to obtain these gains.

In Figure 27.3, we give a comparison of a SISO channel with 2 MIMO channels, (2×2) and (4×4) using equations we will describe later in this chapter. At SNR of 10 dB, a 2×2 MIMO system offers 5.5 b/s/Hz and whereas a 4×4 MIMO link offers over 10 b/s/Hz. This is an amazing increase in capacity without any increase in transmit power! Just by increasing the number of transceivers. Not only that, this superb performance comes in what have always been considered awful channels, those that have fading and Doppler.

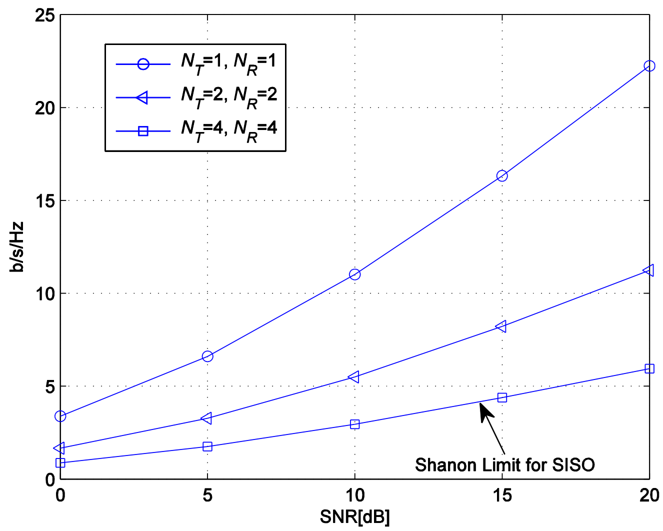


Figure 27.3 – Comparing information theoretic capacity of MIMO systems over single channel systems.

Extending the single link (SISO) paradigm, it is clear that to increase capacity, we can just replicate the link N times. By using N links, we increase the capacity by a factor of N. But this scheme also uses N times the power. Since links are often power-limited, the idea of N link to get N times capacity is not much of a trick. Can we increase the number of links, but not require extra power? How about if we use two antennas but each gets only half the power? This is what is done in MIMO, more transmit antennas but the total power is not increased. Question is how does this result in increased capacity?

The information-theoretic capacity increase under a MIMO is quite large (see equation in Figure 27.4) [3] [4] and easily justifies the increase in complexity. The determination of this increase in capacity and the various parameters that affect this capacity are the subject of this chapter.

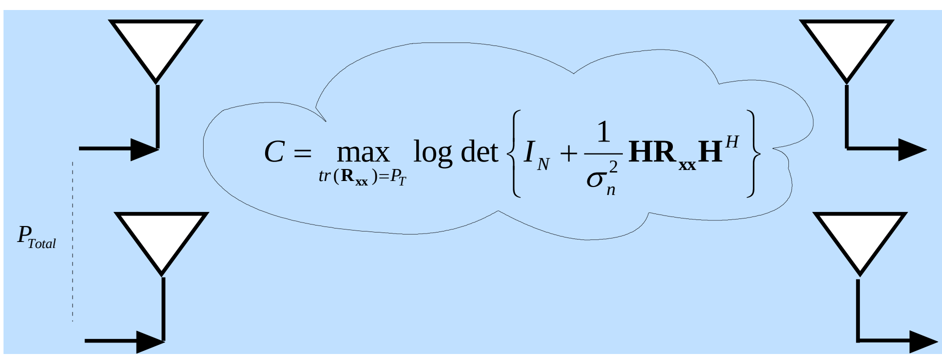


Figure 27.4 - A MIMO channel information theoretic capacity

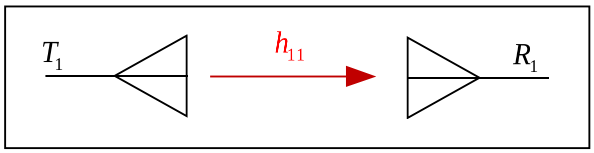
In simple language, MIMO is any link that has multiple transmit and receive antennas. The transmit antennas are collocated, at little less than half a wavelength apart or more. That's approximately 1 cm at 14 GHz and 7.5 cm at 2 GHz. This figure of the antenna separation is determined by mutual correlation function of the antennas using Jakes Model [5]. (See Figure

27.16)The receive antennas are also part of one unit. Just as in SISO links, the communication is assumed to be between one sender and one receiver, although MIMO is also used in multi-user scenario, similar in the way OFDM can be used for one or multiple users.

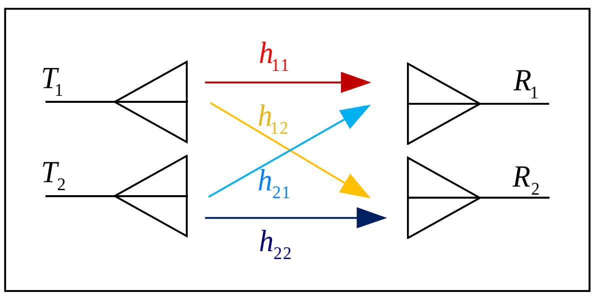
We can write the input/output relationship of a SISO channel as

$$r = hs + n \qquad 27.2$$

where *r* is the received signal, *s* the sent signal and *h*, the impulse response of the channel and *n*, the noise. The term *h*, the impulse response of the channel, can be a gain or a loss, it can be phase shift or it can be time delay, or all of these together. The quantity *h* can be considered an enhancing or distorting agent for the signal SNR.



SISO Channel



MIMO Channel

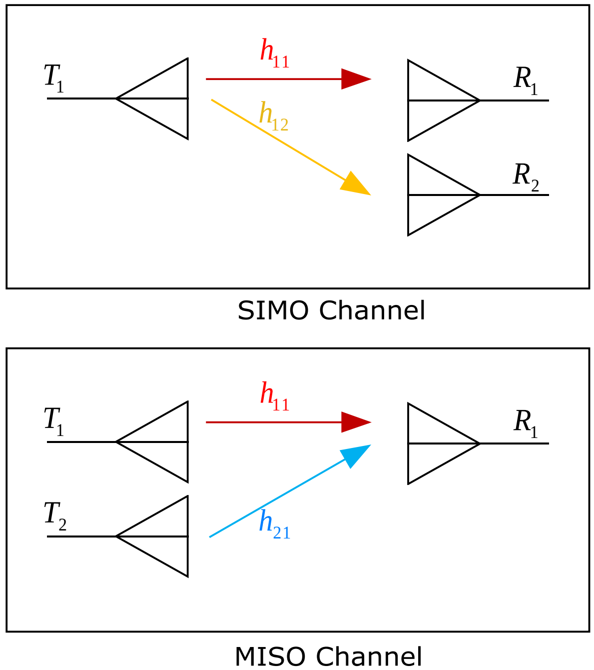


Figure 27.5 – A MIMO channel can be thought of as a matrix channel

Using the same model as SISO, MIMO channel can now be described as

$$\mathbf{R} = \mathbf{HS} + \mathbf{N} \quad (27.3)$$

In this formulation, both transmit and receive signals are vectors. The channel impulse response *h*, is now a matrix, **H**. This channel matrix **H** is called **Channel Information** in MIMO literature.

Dimensionality of Gains in MIMO

The MIMO design of a communications link can be classified in these two ways.

- MIMO using diversity techniques
- MIMO using spatial-multiplexing techniques

Both of these techniques are used together in MIMO systems. With first form, **Diversity technique**, **same data** is transmitted on multiple transmit antennas and hence this increases the *diversity* of the system.

What is diversity? Diversity means that the same data has traveled through *diverse* paths to get to the receiver. Diversity increases the reliability of communications. If one path is weak, then a copy of the data received on another path maybe just fine.

In Figure 27.6, we see a source with data sequence 101 to be sent over a MIMO system with three transmitters. In the diversity form of MIMO, same data, 101 is sent over three different transmitters. If each path is subject to different fading then the likelihood is high that one of these paths will lead to successful reception. This is what we mean by diversity or diversity systems. This system has a diversity gain of 3.

The second form uses **spatial-multiplexing techniques**. In a diversity system, we send same data over each path. Here we multiplex the data 1,0,1 on the three channels. Each channel carries different data, similar to the idea of an OFDM signal. Clearly, by multiplexing the data we have increased the data throughput or the capacity of the channel, but we have lost the diversity gain. The multiplexing has tripled the data rate, so the multiplexing gain is 3 but diversity gain is now 0. Whereas in a diversity system the gain comes in form of increased reliability, here the gain comes in form of increased data rate.

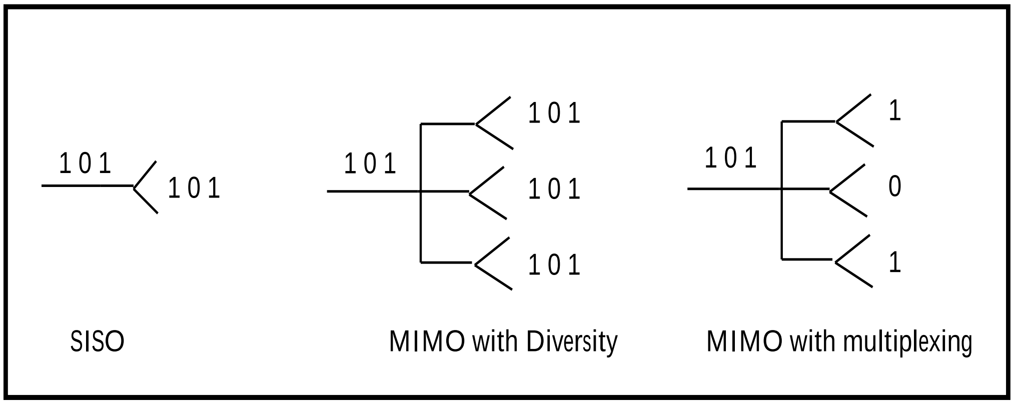


Figure 27.6 – Equivalent MIMO systems; (a) SISO System, (b) MIMO Diversity System, (c) MIMO Multiplexing System

Characterizing a MIMO channel

When a channel uses a multiple of receive antennas, $N_{R,}$ and multiple transmit antennas, $N_{T,}$ it is called a multiple-input, multiple output (MIMO) system.

When $N_T = N_R = 1$, a SISO system.
When $N_T > 1$ and $N_R = 1$, called a MISO system,
When $N_T = 1$ and $N_R > 1$, called a SIMO system.
When $N_T > 1$ and $N_R > 1$, is a MIMO system.

In a typical SISO channel, we transmit the data and we take our chances. As long as we know that the SNR is not changing dramatically, we do not ask any information about the channel on a bit by bit basis. We call this a stable channel. Channel knowledge of a SISO channel is characterized only by its steady-state SNR.

What do we mean by channel knowledge for MIMO channel? Assume that we have a link with two transmitters and two receivers on each side. We transmit the same symbol from each antenna at the same frequency, which is received by two receivers. There are four possible paths as shown in Figure 27.5.

Each path from a transmitter to a receiver has some loss/gain associated with it and we can characterize a channel by this loss. (A path may actually be sum of many multipath components but it is characterized only by the start and the end points.) Since all four channels in this example are carrying the same symbol, this provides diversity by making up for a weak channel, if any. In Figure 27.7 we see how each channel may be fading from one moment to the next. At time tick 32, for example, the fade in channel h21 is much higher than the other three.

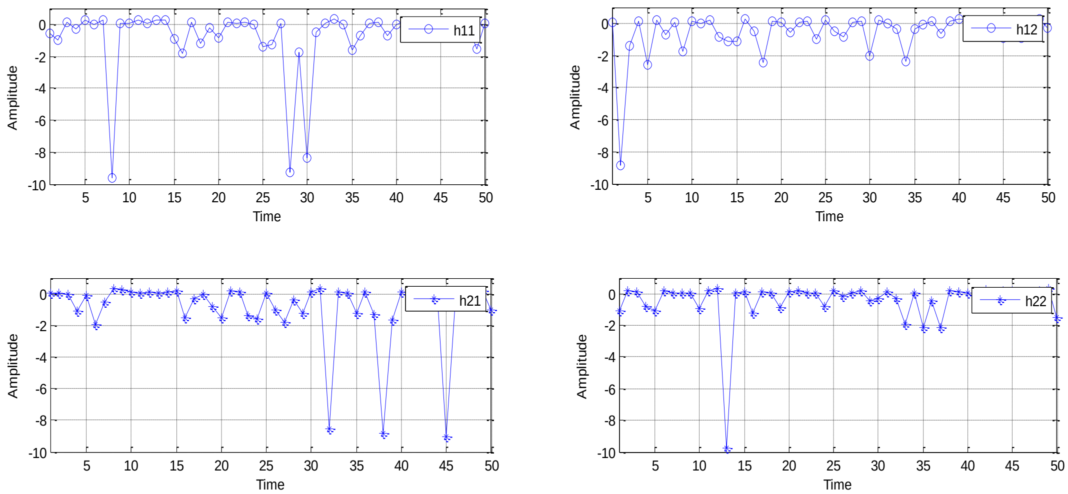


Figure 27.7 – Loss coefficients of the 2x2 MIMO channel over time

As the number of antennas, hence the paths increase in a MIMO system, there is an associated increase in diversity. It does not take a lot of imagination to extrapolate this and see that with increasing numbers of transmitters, we can probably compensate for all fades. With increasing diversity, the fading channel starts to look like a Gaussian channel, which is a very welcome outcome.

The relationship between the received signal in a MIMO system and the transmitted signal can be represented in a matrix form with H matrix representing the low-pass channel response h_{ij}, which is the channel response from the j_{th} antenna to the i_{th} receiver.
(Often the word receiver and the receiving antenna are used as synonyms. Same applies to transmitter and transmitter antenna.) The matrix H of size (N_R, N_T) has N_R rows, representing N_R received signals, each of which is composed of N_T components from N_T transmitters. Each column of the H matrix represents the components arriving from one transmitter to N_R receivers.

The H matrix is called the **<u>channel information</u>**. Each of its entries is a distortion coefficient acting on the transmitted signal amplitude and phase in time-domain.

How is this information developed? A symbol is sent from the first antenna in, and a response is noted by all three receivers. Then the other two antennas do the same thing and a new column is developed by the three new responses.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} .2 & -.6 & .8 \\ -.1 & .8 & .6 \\ .1 & .4 & .2 \end{bmatrix} \tag{27.4}$$

In this example, when a 1 is sent by the first transmitter, each of the 3 receivers sees amplitude values [.2, -.1 , .1]. The process is repeated twice more and the H matrix is complete. Note that the H matrix is developed by the receiver, transmitter typically does not have any idea what the channel looks like. It is transmitting blindly. If the receiver then turns around and transmits this matrix back to the transmitter, then the transmitter would be able to see how the signals are faring and might want to make adjustments in the powers allocated to its antennas. Perhaps a smart computer at the transmitter will decide to not transmit on antenna 1, since the received signals are so much smaller (in amplitude) than the other two antennas. Maybe we should just split the power between antenna 2 and 3 and turn off antenna 1 until the channel improves. This is a good idea and that's just what is done.

The following shows two examples of an H matrix, the first with only amplitude changes and the second with complex entries that include both amplitude and phase changes which is a more realistic scenario.

$$\begin{bmatrix} 0.8 & 0.5 & 0.3 \\ 0.4 & 1.0 & 0.2 \\ 0.5 & 0.5 & 0.6 \end{bmatrix} \qquad \begin{bmatrix} 0.8 & 0.5-j0.2 & 0.3+j0.6 \\ 0.4-j0.6 & 1.0-j0.1 & 0.2-j0.9 \\ 0.5+j0.3 & 0.5+j1.5 & 0.6+j1.2 \end{bmatrix} \qquad 27.5$$

Modeling a MIMO Channel

We start with a general channel which has both multipath and Doppler (the conditions facing a mobile in case of a cell phone system). The channel matrix H for this channel takes this form.

$$H(\tau,t) = \begin{bmatrix} h_{11}(\tau,t) & h_{12}(\tau,t) & \cdots & h_{1N_T}(\tau,t) \\ h_{21}(\tau,t) & h_{22}(\tau,t) & \cdots & h_{2N_T}(\tau,t) \\ \vdots & \vdots & \ddots & \vdots \\ h_{N_R,1}(\tau,t) & h_{N_R,2}(\tau,t) & \cdots & h_{N_R N_T}(\tau,t) \end{bmatrix} \qquad 27.6$$

Each path coefficient is a function of not only time t because the mobile is moving but also a time delay relative to other paths. The variable τ indicates relative delays between each component caused by frequency shifts. The time variable *t* represents the time-varying nature of the channel such as one that has Doppler or other time variations. [6]

If the transmitted signal is $s_i(t)$, and the received signal is $r_i(t)$, we write the input-output relationship of a general MIMO channel as

$$\begin{aligned} r_i(t) &= \sum_{j=1}^{N_T} \int_{-\infty}^{\infty} h_{ij}(\tau,t)\, s_j(t-\tau)\, d\tau \\ &= \sum_{j=1}^{N_T} h_{ij}(\tau,t) * s_j(\tau) \quad i = 1,2 \ldots N_R \end{aligned} \qquad 27.7$$

The channel equation for the received signal $r_i(t)$ is expressed as convolution of the channel matrix H and the transmitted signals because of the delay variable τ . We write this relationship in matrix form as

$$\mathbf{r}(t) = \mathbf{H}(\tau,t) * \mathbf{s}(t) \qquad 27.8$$

If we assume that the channel is flat (non-frequency selective), but is time-varying, i.e. has Doppler, we would write this relationship without the convolution as

$$\mathbf{r}(t) = \mathbf{H}(t)\,\mathbf{s}(t) \qquad 27.9$$

In this case, the H matrix changes randomly with time. If the time variations are very slow (non-moving receiver and transmitter) such that during a block of transmission longer than the several symbols, we can assume the channel to be non-varying, or static. A fixed realization of the H matrix can be written as follows (27.11). The individual entries can be either scalar or complex.

For analysis purposes, we can make some important assumptions about the H matrix. We can assume that it is fixed for a period of one or more symbols and then changes randomly. This is a fast change and causes the SNR of the received signal to change very rapidly. Or we can assume that it is fixed for a block of time, such as over a full code sequence, which makes decoding easier because the decoder does not have to deal with a variable SNR over a block. Or we can assume that the channel is semi-static such as in a TDMA system, and its behavior is static over a burst or more. Each version of the H matrix seen is called its *realization*. How fast these realizations change depends on the channel type.

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1N_T} \\ h_{21} & h_{22} & \cdots & h_{2N_T} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N_R,1} & h_{N_R,2} & \cdots & h_{N_R N_T} \end{bmatrix} \qquad 27.10$$

For a fixed random realization of the H matrix, the input-output relationship can be written without the convolution as

$$\mathbf{r}(t) = \mathbf{H}\,\mathbf{s}(t) \qquad 27.11$$

In this channel model, the H matrix is assumed fixed. An example of this type of situation where the H matrix may remain fixed for a long period would be a phone call taking place from one fixed place to another. In most cases, we can consider the channel to be static. This allows us to treat the channel as deterministic over that period and amenable to analysis.

The power received at all receive antennas is equal to the sum of the total transmit power, assuming channel offers no gain or loss. Each entry h_{ij} is an amplitude and phase term. Squaring it give us the power for that path. There are N_T paths to each receiver, so the sum of j terms, gives us the total transmit power. Each receiver receives the total transmit power. For this relation we have assumed that the transmit power of each transmitter is 1.0.

$$\sum_{j=1}^{N_T} \left(h_{ij}\right)^2 = N_T \qquad 27.12$$

The H matrix is a very important construct in understanding MIMO capacity and performance. How a MIMO system performs depends on the condition of the channel matrix H and its properties. The H matrix can be thought of as a set of simultaneous equations. Each equation represents a received signal which is a composite of unique set of channel coefficients applied to the transmitted signal.

$$r_1 = h_{11}s + h_{12}s \cdots + h_{1N_T}s \qquad 27.13$$

If the number of transmitters is equal to the number of receivers, then there exists a unique solution to these equations. If the number of equations is larger than the number of unknowns (i.e. $N_R > N_T$) then the solution can be found using a zero-forcing algorithm. When $N_T = N_R$, then the solution can be found by (ignoring noise) inverting the H matrix as in

$$\hat{s}(t) = H^{-1} r(t) \qquad 27.14$$

The system performs best when the H matrix is full rank, with each row/column meeting conditions of independence. What this means is that best performance is achieved only when each path is fully independent of all others. This can happen only in an environment that offers rich scattering, fading and multipath, which seems like a counter-intuitive statement. But if we look at the equation above, we see that the only way to extract the transmitted information is when the H matrix is invertible. And the only way it is invertible is if all its rows and columns are uncorrelated, something we learn in linear algebra. And the only way we can have that is if the scattering, fading and all other effects cause the channels to be completely uncorrelated.

Diversity Domains and MIMO Systems

In order to provide a fixed quality of service, a large amount of transmit power is required in a Rayleigh or Rican fading environment to assure that no matter what the fade level, adequate power is still available to decode the signal. Diversity techniques that mitigate multipath fading, both slow and fast are called **Micro-diversity**, whereas those resulting from path loss, from shadowing due to buildings etc. are an order of magnitude slower than multipath, are called **Macro-diversity** techniques. MIMO design issues are limited only to micro-diversity. Macro-diversity is usually handled by providing overlapping base station coverage and handover algorithms and is a separate independent operational issue.

In time domain, repeating a symbol *N* times is the simplest example of increasing diversity. Interleaving is an another example of time diversity where symbols are artificially separated in time so as to create time-separated and hence independent fading channels for adjacent symbols. Error correction coding also accomplishes time-domain diversity by spreading the symbols in time. Such time domain diversity methods are termed **Temporal diversity**.

Frequency diversity can be provided by spreading the data over frequency, such as is done by spread spectrum systems. In OFDM frequency diversity is provided by sending each symbol over a different frequency. In all such frequency diversity systems, the frequency separation must be greater than the coherence bandwidth of the channel in order to assure independence.

The type of diversity exploited in MIMO is called **Spatial diversity.** The **receive side diversity**, is the use of more than one receive antenna. SNR gain is realized from the multiple copies received (because the SNR is additive). Various types of linear combining techniques can take the received signals and use special combining techniques such are Maximal Ratio Combining, Threshold Combing etc. The SNR increase possible via combining results in a power gain. The SNR gain is called the **array gain**. [7]

Transmit side diversity similarly means having multiple transmit antennas on the transmit side which create multiple paths and potential for **angular diversity**. Angular diversity can be understood as beam-forming. If the transmitter has information about the channel, as to where the fading is and which paths (hence direction) is best, then it can concentrate its power in a particular direction. This is an additional form of gain possible with MIMO.

Another form of diversity is **Polarization diversity** such as used in satellite communications, where independent signals are transmitted on each polarization (horizontal vs. vertical). The channels, although at the same frequency, contain independent data on the two polarized hence orthogonal paths. This is also a form of MIMO where the two independent channels create data rate enhancement instead of diversity. So satellite communications is a form of (2, 2) MIMO link.

Related to MIMO but not MIMO

There are some items that bear discussion as they relate to MIMO but are usually not part of it. First are the **smart antennas** used in set-top boxes. Smart antennas are a way to enhance the receive gain of a SISO channel but are different in concept than MIMO. Smart antennas use phased-arrays to track the signal. They are capable of determining the direction of arrival of the signal and use special algorithms such as MUSIC and MATRIX to calculate weights for its phased arrays. They are performing receive side processing only, using linear or non-linear combining.

Rake receivers are a similar idea, used for multipath channels. They are a SISO channel application designed to enhance the received SNR by processing the received signal along several "fingers" or correlators pointed at particular multipaths. This can often enhance the received signal SNR and improve decoding. In MIMO systems Rake receivers are not necessary because MIMO can actually simplify receiver signal processing.

Beamforming is used in MIMO but is not the whole picture of MIMO. It is a method of creating a custom radiation pattern based on channel knowledge that provides antenna gains in a specific direction. Beam forming can be used in MIMO to provide further gains when the transmitter has information about the channel and receiver locations.

Importance of Channel State Information

We will mention the H matrix a lot from now on, since it is at the heart of how MIMO works. We will be calling it by various names, such channel state, channel state information etc. In general we will assume that the receiver is able to get the channel information easily and continuously. It is not equally feasible for the transmitter to obtain a fresh version of the channel state information, because the information has gotten stale on the trip back. However, as long as the transit delay is less than channel coherence time, the information sent back by the receiver to the transmitter retains its freshness and usefulness to the transmitter in managing its power. At the receiver, we refer to channel information as Channel State (or side) Information at the Receiver, **CSIR**. Similarly when channel information is available at the transmitter, it is called **CSIT**. CSI, the channel matrix can be assumed to be known instantaneously at the receiver or the transmitter or both. Although in short term the channel can have a non-zero mean, it is assumed to be zero-mean and uncorrelated on all paths. When the paths are correlated, then clearly, we have less information to exploit. But we can still make the channel work.

Channel information can be extracted by monitoring the received gains of a known sequence. In Time Division Duplex (TDD) communications where both transmitter and the receiver are on the same frequency, the channel condition is readily available to the transmitter. In Frequency Division Duplex (FDD) communications, since the forward and reverse links are at different frequencies, this requires a special feedback link from the receiver to the transmitter. In fact receive diversity alone is very effective but it places greater burden on the smaller handheld receivers, requiring larger weight, size and complex signal processing hence increasing cost.

Transmit diversity is easier to implement from a system point of view because the base station towers in a cell system are not limited by power or weight. In addition to adding more transmit antennas on the base station towers, space-time coding is also used by the transmitters. This makes the signal processing required at the receiver simpler.

MIMO Gains

Our goal is to transmit and receive data over several independently fading channels such that the composite performance mitigates deep fades on any of the channels. To see how MIMO enhances performance in a fading or multipath channel, we first examine the BER for a BPSK signal as a function of the receive SNR. [8; 7]

$$P_e \approx Q\left(\sqrt{2\|h\|^2 \, SNR}\right) \qquad 27.15$$

We use here a general expression of SNR instead of the quantity Es/N0. The quantity $\left(|h|_i^2 \times SNR\right)$ is the instantaneous SNR over the i_{th} path determining the BER for that path.

Now assume that there are L possible paths, where $L = N_R \times N_T$, with N_T = number of transmitter and N_R = number of receive antennas. Since there are several paths, the average BER can be expressed as a function of the average channel gain over all these paths, which we write as $\|h\|^2$. This quantity is the average gain over all channels, L.

$$\|h\|^2 = Avg\left[|h_l|^2\right] = \sum_{l=1}^{L} |h_l|^2 \qquad 27.16$$

We can rewrite the average SNR as a product of two terms.

$$\|h\|^2 \, SNR = \underline{L \times SNR} \cdot \underline{\frac{1}{L}\|h\|^2} \qquad 27.17$$

The first part on the LHS, $(L \times SNR)$ is a linear increase in SNR due to the L paths. This term $(L \times SNR)$ is called by various names; **<u>power gain, rate gain or array gain</u>**. This term can also include beamforming gain. Hence increasing the number of antennas increases the array gain directly by the factor L. [7]

The second term $\frac{1}{L}\|h\|^2$ is called **diversity gain**. This is the average gain over L different paths. It seems intuitive that if one of the paths exhibits deep fading then, when averaged over a number of independent paths, the deep fades can be averaged out. (We use the term channel to mean the composite of all paths.) This is akin to the greater reliability of striking a target with shotgun pellets compared to a single bullet. Hence on the average we would experience a diversity gain as long as the path gains across the channels are not correlated. If the gains are correlated, such as if all paths are mostly line-of-sight, we would obtain only an array gain and very little diversity gain. This is intuitive because a diversity gain can come only if the paths are diverse, or in other words uncorrelated.

MIMO Advantages

Operating in Fading Channels

The most challenging issue in communications signal design is how to mitigate the effects of fading channels on the signal BER. A fading channel is one where channel gain is changing dramatically, even at high SNR, and as such it results in poor BER performance as compared to an AWGN channel. For communications in a fading channel, we want a way to convert the highly variable fading channel to a stable AWGN-like channel.

Multipath fading is a phenomenon that occurs due to reflectors and scatters in the path. The measure of multipath is **Delay Spread**, which is the RMS time delay as a function of the power of the multipath. This delay is converted to a **Coherence Bandwidth (CB)**, a often used metric of multipath. Remember that a time delay is equivalent to a frequency shift in the frequency domain. So any distortion that delays a signal, changes its frequency. So delay spread -> bandwidth distortion.

Whether a signal is going through a flat or a frequency-selective fading at any particular time is a function of coherence bandwidth of the channel as compared with its bandwidth as shown in Table I. If the Coherence bandwidth of the channel is larger than the signal bandwidth, then we have a flat or a non-frequency selective channel. What coherence means is that all the frequencies in the signal respond similarly or are subject to the same amplitude distortion. This means that fading does not affect frequencies differentially, which is a good thing. Differential distortion is hard to deal with. So of all types of fading, flat fading is the least problematic.

The next source of distortion is Doppler. You know all about the train, and its whistle and know that Doppler is a function of direction of the signal, hence depending on what and where, Doppler results in different distortions to the frequency band of the signal. The measure of **Doppler spread** is called **Coherence Time (CT)** (*no relationship to Coherence Bandwidth from the flat-fading case*). The comparison of the CT with the symbol time determines the speed of fading. So if the coherence time is very small, compared to the symbol time, that's not good. [9]

The idea of Coherence Time and Coherence Bandwidth is often confused. How to remember which metric applies to which type of fading? Remember that flatness refers to frequency response and not to time. So Coherence Bandwidth determines whether a channel is considered flat or not.

Coherence Time, on the other hand has to do with changes over time, which is related to motion. Coherence Time is the duration during which a channel appears to be unchanging. So think about Coherence Time when Doppler or motion is present. When Coherence Time is longer than symbol time, then we have a slow fading channel and when symbol time is longer than Coherence time, then we have a fast fading channel. So slowness and fastness mean time based fading.

Table I – MIMO channel Types and their Measures

Channel Spread	Channel Selectivity	Type	Measure
Delay Spread	Frequency	Non-selective	Coherence Bandwidth > Signal Bandwidth
	Frequency	Selective	Signal Bandwidth > Coherence Bandwidth
Doppler Spread	Time	Slow-fading	Coherence Time > Symbol Time
		Fast-fast fading	Symbol Time > Coherence Time
Angle Spread	Beam pattern	-	Coherence Distance

In addition to these *fast* channel effects, we also have mean path loss as well as rain losses, which are considered order-of-magnitude slower effects and are managed operationally and so will not discuss them as part of the advantages of MIMO.

How MIMO creates performance gains in a fading channel

Shannon defines capacity of a channel as a function of its SNR. Underlying this is the assumption that the SNR is invariant. For such a system, Shannon capacity is called its **ergodic capacity**. Since SNR is related to BER, the capacity of a channel is directly related to how fast the BER declines with SNR. We want the BER to decrease quickly with increasing power.

The Rayleigh channel BER when compared to an AWGN channel for the same SNR is considerably bigger and hence the capacity of a Rayleigh channel which we can think of as the converse of its BER, is much lower. Using the BER of a BPSK signal as a benchmark, we examine the shortfall of a Rayleigh channel and see how MIMO can help to mitigate this loss.

For A BPSK signal, the BER in an AWGN channel is given by (setting $\|h\|^2 = 1$ in (27.16))

$$p_e = Q(\sqrt{2SNR}) \qquad 27.18$$

The BER of the same channel in a Rayleigh channel is given by [8; 8]

$$p_e = \frac{1}{2}\left(1 - \sqrt{\frac{SNR}{1+SNR}}\right) \approx \frac{1}{2SNR} \qquad 27.19$$

Figure 27.8 shows the BER of an AWGN and a Rayleigh channel as a function of the SNR. The AWGN BER varies by the inverse of the square of the SNR, SNR^{-2} and declines much faster than the Rayleigh channel which declines instead by SNR^{-1}. Hence an increase in SNR helps the Rayleigh channel much less than it does an AWGN channel.

We can say that Rayleigh channel improves much more slowly as more power is added.

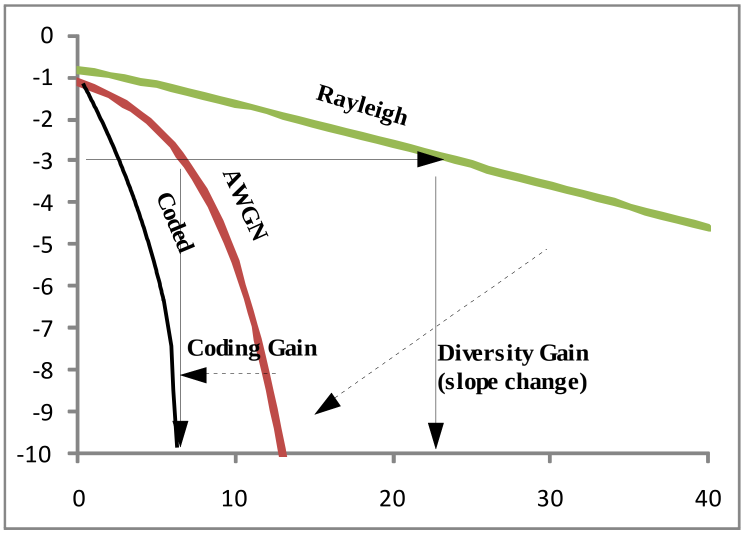


Figure 27.8 – BER declines as function of the exponent of the SNR.

In Figure 27.8, we see that for a BER of 10^{-3}, an additional 17 dB of power is required in a Rayleigh channel. This is a very large differential, nearly 50 times larger than AWGN. One way to bring the Rayleigh curve closer to the AWGN curve (which forms a limit of performance) is to add more antennas on the receive or the transmit-side hence making SISO into a MIMO system.

Starting with just one antenna, let's increase the number of receive antennas to N_R, while keeping one transmitter, making it a SIMO system. Assuming optimum combining of the two received signals, or Maximal Ratio Combining (MRC), we write the relationship of BER [8] of a BPSK signal under a fading channel with N receive antennas, as

$$BER = \left(\frac{1-\mu}{2}\right)^{N_R} \sum_{l=0}^{N_R-1} \binom{N-1-l}{l} \left(\frac{1+\mu}{2}\right)^{l}, \quad \mu = \sqrt{\frac{SNR}{1+SNR}} \qquad 27.20$$

The asymptotic BER at large SNR (large SNR has no formal definition, anything over 15 dB can be considered large.) is given as an approximation as [8]

$$BER(N, SNR) = \left(\frac{1}{4SNR}\right)^{N_R} \binom{2N-1}{N} \qquad 27.21$$

To determine what we gain by adding just one more antenna to the N antennas, we take the ratio of the current BER to the BER due to one more antenna.

$$\frac{BER(N, SNR)}{BER(N+1, SNR)} = SNR\left(1 + \frac{1}{2N+1}\right) \quad 27.22$$

The gain from adding one more antenna is equal to SNR multiplied by a delta increase in SNR. The delta increase diminishes as more and more antennas are added. The largest gain is seen when going from a single antenna to two antennas, (1.5 for going from 1 to 2 vs. 1.1 for going from 4 to 5 antennas). This delta increase is similar in magnitude to the slope of the BER curve at large SNR.

Formally, a parameter called **Diversity order** d, is defined as the slope of the BER curve as a function of SNR in the region of high SNR.

$$d = -\lim_{SNR \to \infty} \log \frac{BER(SNR)}{SNR} \quad 27.23$$

Figure 27.9 shows the gains possible with MIMO as more receive antennas are added. As more and more antennas are added, a Rayleigh channel approaches the AWGN channel.

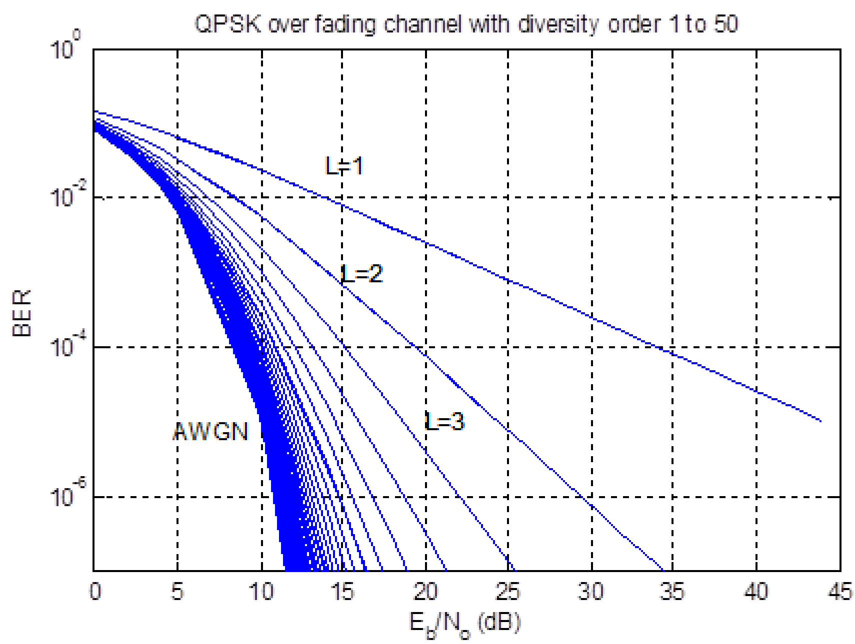


Figure 27.9 – Diversity Gain in a fading channel

Should we keep on increasing the number of antennas indefinitely? No, beyond a certain number, increase in number of paths, L does not lead to significant gains. When complexity is taken into account, a small number of antennas is enough for satisfactory performance.

Capacity of MIMO channels

Capacity of a SISO Channel

All system designs strive for a target capacity of throughput. For SISO channels, the capacity is calculated using the well-known Shannon equation. Shannon defines capacity for an ergodic channel that data rate which can be transmitted with asymptotically small probability of error. The capacity of such a channel is given in terms of bits/sec or by normalizing with bandwidth by bits/sec/Hz. The second formulation (27.26) allows easier comparison and is the one used more often. It is also bandwidth independent.

$$C = W\log_2(1+\frac{P}{N_0 W}) \quad b/s \qquad 27.24$$

$$= \log_2(1+SNR) \quad b/s/Hz \qquad 27.25$$

At high SNRs, ignoring the addition of 1 to SNR, the capacity is a direct function of SNR.

$$C = \log_2(SNR) \qquad 27.26$$

This capacity is based on a constant data rate and is not a function of whether channel state information is available to the receiver or the transmitter. This result is applicable only to ergodic channels, ones where the data rate is fixed and SNR is stable.

Capacity of MIMO Channels

We know from Shanon's equation that a particular SNR can give only a fixed maximum capacity. If SNR goes down, so will the ability of the channel to pass data. In a fading channel, the SNR is constantly changing. As the rate of fade changes, the capacity changes with it.

We can use a fixed H matrix as our benchmark of performance where the basic assumption is that, for that one realization, the channel is fixed and hence has an ergodic channel capacity. In other words, for just that little time period, the channel is behaving like an AWGN channel. We then break a channel into portions of either time or frequency so that in small segments, even in a frequency-selective channel with Doppler, channel can be treated as having a fixed realization of the H matrix, i.e. allowing us to think of it instantaneously as a AWGN channel. We can perform the capacity calculations over several realizations of H matrix and then compute average capacity over these. In flat fading channels the channel matrix may remain constant and or may change very slowly. However, with user motion, this assumption does not hold.

Before delving into capacity calculations, we will look at how a MIMO matrix channel can be decomposed into parallel independent channels. This method provides an alternate way to look at the capacity of a MIMO system.

Decomposing a MIMO channel into parallel independent channels

Conceptually we think of MIMO as transmission of same data over multiple antennas, hence it is a matrix channel. But there is a mathematical trick that lets us decompose the MIMO channel into several independent parallel channels each of which can be thought of as a SISO channel. To look at a MIMO channel as a set of independent channels, we use an algorithm that comes from linear algebra, the Singular Value Decomposition (SVD). The process requires **pre-coding** at the transmitter and **receiver shaping** at the receiver. It may look hard to understand but it is just matrix math.

Input and output auto-correlation

Assume that a MIMO channel has N transmitters and M receivers. The transmitted vector across N_T antennas is given by $x_1, x_2, \cdots x_{N_T}$. We assume that individual transmit signals consist of symbols that are zero mean circular-symmetric complex Gaussian variables. (A vector x is said to be circular-symmetric if $e^{j\theta}\mathbf{x}$ has same distribution for all θ.) The covariance matrix for the transmitted symbols is written as

$$\mathbf{R}_{xx} = E\left\{\mathbf{x}\mathbf{x}^H\right\} \qquad 27.27$$

Where symbol H stands for the transpose and component-wise complex conjugate of the matrix (also called Hermitian) and not the channel matrix. This relationship gives us a measure of correlated-ness of the transmitted signal amplitudes.

When the powers of the transmitted symbols are the same, what we get is a scaled identity matrix. For a (3×3) MIMO system of total power of P_T, equally distributed we would write this matrix as

$$\mathbf{R}_{xx} = P_T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If the same system distributes the power differently say in ratio of 1: 2: 3, then the covariance matrix would be

$$\mathbf{R}_{xx} = P_T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

If we assume that the total transmitted power is P_T and is equal to trace of the Input covariance matrix, we can write the total power of the transmitted signal as the trace of the covariance matrix.

$$P = tr\left\{\mathbf{R}_{xx}\right\} \quad 27.28$$

The received signal is given by

$$\mathbf{r} = \mathbf{Hx} + n \qquad 27.29$$

The noise matrix (N×1) components are assumed to be ZMGV (zero-mean Gaussian variable) of equal variance. We can write the covariance matrix of the noise process similar to the transmit symbols as

$$\mathbf{R}_{\mathbf{nn}} = E\left\{\mathbf{nn}^H\right\} \qquad 27.30$$

And since there is no correlation between its rows, we can write this as

$$\mathbf{R}_{nn} = \sigma^2 \mathbf{I}_M \qquad 27.31$$

Which says that each of the M received noise signals is an independent signal of noise variance, σ^2. Each receiver receives a complex signal consisting of the sum of the replicas from N transmit antennas and an independent noise signal.

If we assume that the power received at each receiver is not the same, we write the SNR of the m_{th} receiver as

$$\gamma_m = \frac{P_m}{\sigma^2} \qquad 27.32$$

where P_m is some part of the total power. However the ***average*** SNR for all receive antennas would still be equal to $\frac{P_T}{\sigma^2}$, where P_T is total power because

$$P_T = \sum_{m=1}^{N_T} P_m$$

Now we write the covariance matrix of the receive signal using eq. (27.33) as

$$\mathbf{R_{rr}} = \mathbf{H R_{xx} H}^H + \mathbf{R_{nn}} \qquad 27.33$$

where $\mathbf{R_{xx}}$ is the covariance matrix of the transmitted signal. The total receive power is equal to the trace of the matrix $\mathbf{R_{rr}}$.

Singular Value Decomposition (SVD)

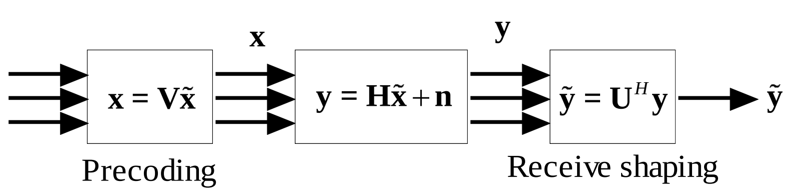


Figure 27.10 – Modal decomposition of a MIMO channel with full CSI

SVD is a mathematical application that lets us create an alternate structure of the MIMO signal. In particular we examine the MIMO signal by looking at the eigenvalues of the H matrix. The H matrix can be written in Singular Value Decomposition (SVD) form as

$$H = U \Sigma V^H \qquad 27.34$$

Where U and V are **unitary matrices** ($U^H U = I_{N_R}$, and $V^H V = I_{N_T}$) and Σ is a $N_R \times N_T$ diagonal matrix of **singular values** (σ_i) of H matrix. If H is a full Rank matrix then we have a

$\min(N_R, N_T)$ of non-zero singular values, hence the same number of independent channels. The parallel decomposition is essentially a linear mapping function performed by pre-coding the input signal $\tilde{x}$, consisting of multiplying it with matrix V, such that $x = V\tilde{x}$.

The received signal $\tilde{\mathbf{y}}$ is given by multiplying it with $\mathbf{U}^H$,

$$\tilde{\mathbf{y}} = \mathbf{U}^H(\mathbf{Hx} + \mathbf{n}) \qquad 27.35$$

Now multiplying it out, and setting value of H from (27.35), we get

$$\tilde{\mathbf{y}} = \mathbf{U}^H(\underline{\mathbf{U\Sigma V}^H}\mathbf{x} + \mathbf{n})$$

Now substitute $x = V\tilde{x}$ into above. We get

$$\begin{aligned}\tilde{\mathbf{y}} &= \mathbf{U}^H(\underline{\mathbf{U\Sigma V}^H}\mathbf{x} + \mathbf{n}) \\ &= \mathbf{U}^H(\mathbf{U\Sigma V}^H V\tilde{\mathbf{x}} + \mathbf{n}) \\ &= \mathbf{U}^H\mathbf{U\Sigma V}^H\mathbf{V}\tilde{\mathbf{x}} + \mathbf{U}^H\mathbf{n} \\ &= \Sigma\tilde{\mathbf{x}} + \tilde{\mathbf{n}}\end{aligned} \qquad 27.36$$

In the last result we see that the output signal is in form of a pre-coded input signal $\tilde{\mathbf{x}}$ times the singular value matrix, Σ. Note that the multiplication of noise **n**, by the unitary matrix U^H does not change the noise distribution. [10]

Example 2
Find a parallel channel model for a MIMO system, the H matrix of which is given by

$$\begin{bmatrix} 0.2 & 0.4 & 0.8 \\ 0.7 & 0.3 & 0.4 \\ 0.5 & 0.7 & 0.2 \end{bmatrix} \qquad 27.37$$

The SVD decomposition obtained using Matlab is given by

$$H = \begin{bmatrix} -0.5774 & -0.7995 & -0.1657 \\ -0.5774 & 0.2563 & 0.7752 \\ -0.5774 & 0.5432 & -0.6096 \end{bmatrix} \begin{bmatrix} 1.4 & 0 & 0 \\ 0 & 0.5359 & 0 \\ 0 & 0 & 0.3359 \end{bmatrix} \begin{bmatrix} -0.5774 & 0.5432 & 0.6096 \\ -0.5774 & 0.2563 & -0.7752 \\ -0.5774 & -0.7995 & 0.1657 \end{bmatrix} \quad 27.38$$

The center matrix contains the **singular values**, σ_i of the H matrix. This is the Σ matrix. The number of singular values is equal to the rank of the matrix. This process decomposes the matrix channel into three independent channels, with gains of 1.4, 0.5359 and 0.3359 respectively.

The input signal in this case would be first multiplied by V matrix, and the output signal would be multiplied by the inverse of the U^H matrix.

The three channels characterized by the three singular values can be treated as SISO channels, however with different gains. The first channel with the gain of 1.4 will have better performance than the other two. In Figure 27.11 the decomposition is shown as three different channels.

Important thing to note: the only way SVD can be used is if the transmitter knows what pre-coding to apply, which of course requires knowledge of the channel by the transmitter.

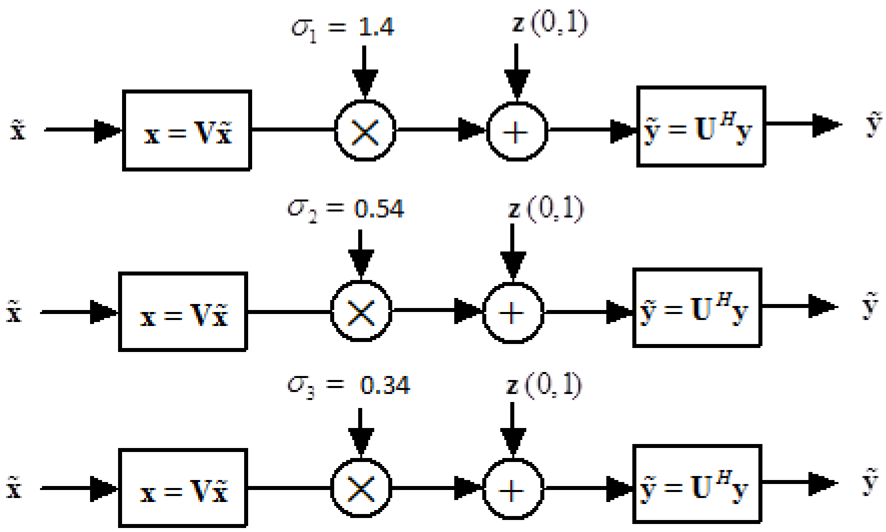


Figure 27.11 – SVD decomposes a matrix channel into parallel equivalent channels.

One might rightfully ask, "Since SVD entails greater complexity, not the least of which is feeding back CSI to the transmitter, with the same results, why should we consider the SVD approach?" The answer is that the SVD approach allows the transmitter to optimize its distribution of transmitted power, thereby providing a further benefit – *transmit array gain*.

The channel *eigenmodes* (or principle components) can be viewed as individual channels characterized by coefficients (*eigenvalues*). The number of significant eigenvalues specifies the maximum degree of diversity. The larger a particular eigenvalue, the more reliable is that channel. The principle eigenvalue specifies the maximum possible beamforming gain. The most important benefit of the SVD approach is that it allows for enhanced array gain – the transmitter can send more power over the better channels, and less (or no) power over the worst ones. The number of principle components is a measure of the maximum degree of diversity that can be realized in this way.

Channel capacity of a SIMO, MISO channel

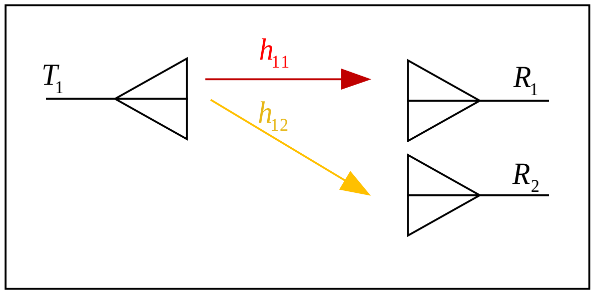


Figure 27.12 A single in–multiple out, SIMO channel

Before we go on to discuss the capacity of a MIMO channel, let's examine the capacity of a channel that has multiple receivers or transmitters but not both. When there is only one transmitter and multiple receivers, the capacity of the SIMO channel is a modification of (27.15) given by the expression in [7].

We modify the SNR of a SISO channel by the gain factor obtained from having multiple receivers.

$$C_{SIMO} = \log_2\left(1 + \|\mathbf{h}\|^2 \, SNR\right) \quad \text{bits/s/Hz} \qquad 27.39$$

Where the term $\|\mathbf{h}\|^2$ is equal to $\left(h_1^2 + h_2^2 \cdots + h_{N_R}^2\right)$. The channel consists of only N_R paths and hence the channel gain is constrained by

$$\|\mathbf{h}\|^2 = N_R \qquad 27.40$$

Substituting (27.40 into (27.39) gives the ergodic capacity of the SIMO channel as

$$C_{SIMO} = \log_2\left(1 + N_R \, SNR\right) \quad \text{bits/s/Hz} \qquad 27.41$$

So we are basically increasing the SNR by a factor of N_R. This is a logarithmic gain. Note that we are assuming that the transmitter has no knowledge of the channel.

Let's now consider a MISO channel, with multiple transmitters but one receiver. This does seem like a ridiculous idea, but it is like both mom and dad calling for the child. The effect is better than one doing it!

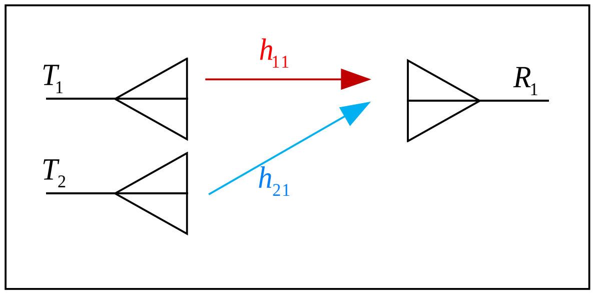


Figure 27.13 A multiple in–single out, MISO channel

The channel capacity of a MISO channel is given by

$$C_{MISO} = \log_2\left(1+\frac{\|\mathbf{h}\|^2 SNR}{N_T}\right) \quad \text{bits/s/Hz} \qquad 27.42$$

Where $\|\mathbf{h}\|^2$ is equal to $\left(h_1^2 + h_2^2 \cdots + h_{N_T}^2\right)$. Why are we dividing by N_T? Compared to the SIMO case, where each path has SNR based on total power, in this case, total power is divided by the number of transmitters. So the SNR at the one receiver keeps getting smaller as more and more transmitters are added. You can think of it this way for a two receiver case; each path has a half of the total power. But since there is only one receiver, this is being divided by the total noise power at the receiver, so the SNR is effectively cut in half.

Again if the transmitter has no knowledge of the channel, the equation devolves in to a SISO channel, because $\|\mathbf{h}\|^2 = N_T$ and Equation (27.42) becomes

$$C_{MISO} = \log_2\left(1+SNR\right) \quad \text{bits/s/Hz} \qquad 27.43$$

The capacity of a MISO channel is less than a SIMO channel when the channel in unknown at the transmitter. However, if the channel is known to the transmitter, then it can concentrate its power into one channel and the capacity of SIMO and MISO channel becomes equal under this condition.

Both SIMO and MISO can achieve diversity but they cannot achieve any multiplexing gains. This is obvious for the case of one transmitter, (SIMO). In a MISO system all transmitters would need to send the same symbol because a single receiver would have no way of separating the different symbols from the multiple transmitters. The capacity still increases only logarithmically with each increase in the number of the transmitters or the receivers. The capacity for the SIMO and MISO are the same. Both channels experience array gain of the same amount but fall short of the MIMO gains.

Capacity of a Constant MIMO channel

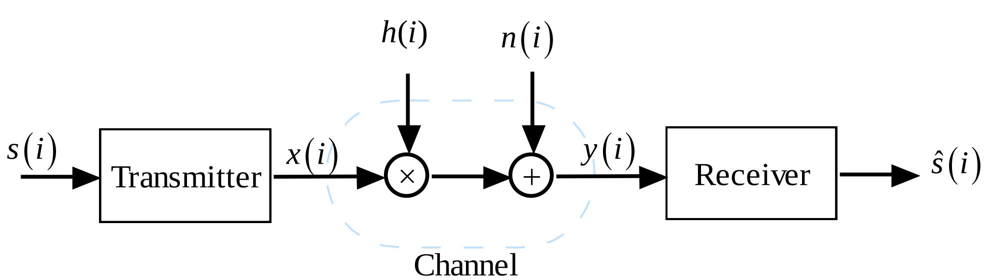


Figure 27.14 - System Channel Model

Let's assume a discrete MIMO channel model as shown in Figure 12. The channel gain maybe time-varying but we assume that it is fixed for a block of time. We also assume that it is random. Assume that total transmit power is P, bandwidth is B and the PSD of noise process is $N_0/2$. Assume that total power is limited by the relationship

$$E\left(\mathbf{x}^H\mathbf{x}\right) = \sum_{i=1}^{N_T} E\left\{\left|x_i\right|^2\right\} = N_T \qquad 27.44$$

The instantaneous SNR, given by $\gamma(i)$ is equal to $P\left|h(i)\right|^2 / N_0 B$. Here h_i is the gain of the i_{th} channel.

We write the input covariance matrix as $\mathbf{R_X} = \mathbf{E}\left[\mathbf{xx^H}\right]$. The trace of this matrix is equal to $\mathrm{Tr}(\mathbf{R_x}) = \rho$ or power per path. When the powers are uniformly distributed (equal) then this is equal to a unity matrix. The covariance matrix of the output signal would not be unity as it is a function of the H matrix.

Now we will develop the capacity expression for a MIMO matrix channel using a fixed but random realization the H matrix. We assume availability of CSIR. The capacity of a deterministic channel is defined by Shanon as

$$C = \max_{f(x)} I(x;y) \text{ bits/channel use} \qquad 27.45$$

I(x;y) is called the mutual information of x and y. The capacity of the channel is the maximum information that can be transmitted from x to y by varying the channel PDF, f(x), the probability density function of the transmit signal x. From information theory we get the relationship of mutual information between two random variables as a function of their differential entropy

$$I(\mathbf{x};\mathbf{y}) = H(\mathbf{y}) - H(\mathbf{y}\,|\,\mathbf{x}) \qquad 27.46$$

The second term is constant for a deterministic channel because it is function only of the noise. So mutual information is maximum only when the term H(y), called differential entropy is maximum.

The differential entropy H(y) is maximized when both x and y are zero-mean, Circular-Symmetric Complex Gaussian (ZMCSCG) random variable. Also from information theory, we write the following relationships.

$$\begin{aligned} H(y) &= \log_2\left\{\det(\pi e R_{yy}\right\} \\ H(y\,|\,x) &= \log_2\left\{\det(\pi e N_0 I_{N_R}\right\} \end{aligned} \qquad 27.47$$

Equations 24.46 and 27.47 should be accepted at faith as they require understanding of information theory. Let's not dwell on them too much. Now we write the signal y as

$$\mathbf{y} = \sqrt{\gamma}\mathbf{Hx} + \mathbf{z} \qquad 27.48$$

Here γ is instantaneous SNR. The auto-correlation of the output signal **y** which we need for (27.48) is given by

$$
\begin{aligned}
R_{yy} &= E\left\{\mathbf{y}\mathbf{y}^H\right\} \\
&= E\left\{\left(\sqrt{\gamma}\mathbf{H}\mathbf{x}+\mathbf{z}\right)\left(\sqrt{\gamma}\mathbf{x}^H\mathbf{H}^H+\mathbf{z}^H\right)\right\} \\
&= E\left\{\left(\gamma\mathbf{H}\mathbf{x}\mathbf{x}^H\mathbf{H}^H+\mathbf{z}\mathbf{z}^H\right)\right\} \\
&= \gamma\mathbf{H}E\left\{\mathbf{x}\mathbf{x}^H\right\}\mathbf{H}^H+E\left\{\mathbf{z}\mathbf{z}^H\right\} \\
&= \gamma HR_{xx}H^H+N_0I_{N_R}
\end{aligned}
$$

27.49

From here we can write the expression for capacity as

$$C = I(x;y) = \max_{Tr(R_{xx})=N_T} \log_2 \det\left\{I_{N_R} + \frac{SNR}{N_T}HR_{xx}H^H\right\} \qquad 27.50$$

When CSIT is not available, we can assume equal power distribution among the transmitters, in which case R_{xx} is an identity matrix and the equation becomes

$$C = \log_2 \det\left(I_{N_R} + \frac{SNR}{N_T}\mathbf{H}\mathbf{H}^H\right) \qquad 27.51$$

This is the capacity equation for MIMO channels with equal power. (Figure 27.3) The optimization of this expression depends on whether or not the CSI (H matrix) is known to the transmitter.

Now note that as the number of antennas increases, we get

$$\lim_{N\to\infty}\frac{1}{M}\mathbf{H}\mathbf{H}^H = \mathbf{I}_N \qquad 27.52$$

Intuitively this means that as the number of paths goes to infinity, the power that reaches each of the infinite number of receivers becomes equal and the channel now approaches an AWGN channel.

This gives us an expression about the <u>capacity limit</u> of a $N_T \times N_R$ MIMO system by substituting (27.52) into (27.51),

$$C = \underline{\underline{M}}\log_2\det\left(I_{N_R} + SNR\right)$$

where M is the minimum of N_T and N_R, the number of the antennas. So now finally we see how the capacity increases linearly with M, the minimum of (N_T, N_R). This is an important finding. If a system has (4, 6) antennas, then the maximum diversity that can be obtained is of order 4, the small number of the two system parameters.

Example 3
Given the following (3×3 MIMO) channel, find the capacity of this channel, given CSIR, no CSIT, SNR = 10 dB and bandwidth equal to 1 kHz. Compare this capacity calculation to that using SVD.

$$H = \begin{bmatrix} 0.4508 & 0.5711 & 0.3450 \\ -0.2097 & 0.4704 & 0.4510 \\ -0.6134 & -0.6382 & -0.4621 \end{bmatrix}$$

Solution:

$$\begin{aligned} C &= B\log 2(\det(I_3 + \frac{SNR}{3}\mathbf{HH}^H) \\ &= 3.798\ kbps \end{aligned} \qquad 27.53$$

The singular values σ_i are equal to: 1.3520, 0.5327, 0.0498.

The SNR for the channels are equal to $\gamma_i = 10\sigma_i^2$

The sum of the capacity of the three independent channels is equal to the same quantity as above equation.

$$B \cdot (\log_2(1 + 1.352^2 \cdot 3.33) + \log_2(1 + .5327^2 \cdot 3.33) + \log_2(1 + .0498^2 \cdot 3.33))$$
$$= 3.798\ kbps$$

If we ignore the third channel and equally distribute the power to the first two channels, the capacity increases to 4.616 kbps. Clearly this is a better way to go but as you can see it requires that transmitter know the condition of the channels.

$$B \cdot (\log_2(1 + 1.352^2 \cdot 5.0) + \log_2(1 + .5327^2 \cdot 5.50))$$
$$= 4.616\ kbps$$

Channel known at transmitter

We can use the SVD results to determine how to allocate powers across the transmitters to get maximum capacity. As we see in example 3, by allocating the power non-equally we can actually increase the capacity. In general, we can say that channels with high SNR (high σ_i), should get more power than those with lower SNR.

There is a solution to the power allocation problem at the transmitter called the water-filling algorithm. This solution is given by

$$\frac{P_i}{P} = \begin{cases} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i}\right) & \gamma_i > \gamma_0 \\ 0 & \gamma_i \le \gamma_0 \end{cases} \qquad 27.54$$

Where γ_0 is a threshold constant. Here γ_i is the SNR of the i_{th} channel.
We are comparing the inverse of the threshold with the inverse of the channel SNR. If the inverse difference is less than the threshold, we do not allocate any power to the i_{th} channel. If the difference is positive then we say, "Hay this channel has life, let's give it some more power to see if it helps the overall performance."

The capacity using the water-filling algorithm is given by

$$C = \sum_{i:\gamma_i > \gamma_0} B \log_2 \left(\frac{\gamma_i}{\gamma_0} \right) \qquad 27.55$$

The thing about water-filling algorithm is that it is much easier to comprehend then is it to describe using equations. Think of it as a boat sinking in the water. Where would you sit on the boat while waiting for rescue, clearly the part that is sticking above the water, right? The analogous part above the surface are the channels that can overcome fading. Some of the channels reach the receiver with enough SNR for decoding. So our data/power should go to these channels and not to the ones that are under water. So basically, we allocate power to those channels that are strongest or above a pre-set threshold. To weak does not go the spoils!

Example 4
Find the optimum power allocation for the MIMO system of Example 3 assuming total power is 1 W, noise power is equal to 0.1 W and the signal bandwidth is 50 kHz.

The singular values computed for the three channels in Example 2 are $\sigma_1 = 1.4$, $\sigma_2 = .5359$ and $\sigma_3 = .3359$. The SNR values for each channel assuming equal power allocation are

$$\gamma_1 = (1/.1)(1.4)^2 = 19.6$$

$$\gamma_2 = (1/.1)(.5359)^2 = 2.87$$

$$\gamma_3 = ((1/.1)(.3359)^2 = 1.128$$

We compute the threshold level from (27.54) to get

$$\sum_{i=1}^{3} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i} \right) = 1$$

$$\frac{3}{\gamma_0} = 1 + ((1/19.6) + (1/2.87) + (1/1.128))$$

$$\gamma_0 = 1.3126$$

Since the third channel with its SNR of 1.128 is less than this threshold value of SNR, we do not allocate any power to the third channel and redo the calculations based only on the first two channels. Repeating the calculations for the two channels with higher singular values, we get a new threshold value of γ_0 = 1.4294. Both channels are above this level, so we should allocate proportional power to each. The power allocated to each channel according to the water-filling algorithm is

$$\frac{P_i}{P} = \frac{1}{1.26} - \frac{1}{\gamma_i}$$
$$P_1 = 1(.793 - (1/9.73)) = 0.691W$$
$$P_2 = 1(.793 - (1/1.875)) = 0.261W$$
$$P_3 = 1(.793 - (1/1.343)) = 0.0491W$$

The total capacity is now equal to

$$C = 50 \times 10^3 \log_2\left(\frac{9.75 + 1.875 + 1.343}{1.26}\right) = 180\ k\text{bits/sec}$$

The allocation has changed from 0.33 W for each transmitter to almost twice that for the first transmitter since it has the best gain. The capacity has increased from 41.4 kbps to nearly five times that.

Channel Capacity in Outage

The Rayleigh channels go through such extremes of SNR fades that the average SNR cannot be maintained from one time block to the next. Due to this, they are unable to support a constant data rate. A Rayleigh channel can be characterized as a binary state channel; an ergodic channel but with an outage probability. When it has a SNR that is above a minimum threshold, it can be treated as **ON** and capacity can be calculated using the information-theoretic rate. But when the SNR is below the threshold, the capacity of the channel is zero. The channel is said to be in outage.

Although ergodic capacity can be useful in characterizing a fast-fading channel, it is not very useful for slow-fading, where there can be *outages* for significant time intervals. When there is an outage, the channel is so poor that there is no scheme able to communicate reliably at a certain fixed data rate.

The outage capacity is the capacity that is guaranteed with a certain level of reliability. We define outage capacity as the information rate that is guaranteed for (100 – *p*)% of the channel realizations. A 1% outage probability means that 99% of the time the channel is above a threshold of SNR and can transmit data. For real systems, *outage capacity* is the most useful measure of throughput capability.

Question: which would have higher capacity, a system with 1% outage or 10% outage?
The high probability of outage means, we can set the threshold lower, which also means that system will have higher capacity, of course only while it is working which is 90% of the time.

We can write the capacity equation of a Rayleigh channel with outage probability ε, as

$$C_{out} = (1 - P_{out}) B \log_2(1 + \gamma_{\min}) \qquad 27.56$$

Where

$$P_{out} = p(\gamma < \gamma_{\min}) \qquad 27.57$$

We can calculate the probability of obtaining a minimum threshold value of the SNR, assuming it has a Rayleigh distribution.

$$P(\gamma < \gamma_{\min}) = \int_0^{\gamma} \frac{1}{\gamma_{\min}} e^{-x/\gamma_{\min}} dx$$
$$P_{out} = 1 - e^{-\frac{\gamma}{\gamma_{\min}}} \qquad 27.58$$

The capacity of channel under outage probability ε is given by

$$C_{\varepsilon} = \log_2\left(1 + SNR \cdot \ln\left(\frac{1}{1-\varepsilon}\right)\right) \qquad 27.59$$

Note that here we have modified the Shanon's equation by the outage probability, the factor in blue.

Example 5
If the average received power of a Rayleigh channel is 20 dBm, then what is the probability that the received power at any time will be less than 2 dBm?

Solution: $\bar{P} = 20\,\text{dbm}$ = 100 mW. The probability that the received power is less than 2 dBm is equal to

$$P_{out} = 1 - e^{-\frac{\gamma}{\lambda_{\min}}} = 1 - e^{-\frac{1.584}{100}} = 0.015 = 15\%$$

In this figure, we plot the capacity as a function of the outage probability for MIMO systems.

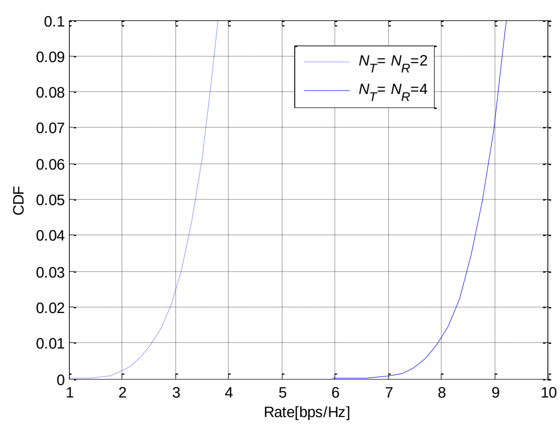


Figure 27.15 – System capacity as a function of outage probability. Low probability means low capacity. We can increase the number of antennas to increase capacity for a given outage probability.

Example 6

Assume a fading channel which can take on three different values of channel coefficients $h(i)$: 0.4 with probability 0.2, 0.1 with probability 0.5, and 0.2 with probability 0.3. If the transmit power is 10mW, the noise density $N_0 = 10^{-9}$ W/Hz and the bandwidth of the signal is equal to 50 kHz, find capacity of this fading channel and the capacity of an equivalent AWGN channel of the same average SNR.

Solution:

The three SNR values are equal to Ph_i^2 / N_0W.

$$= .01\times(.4)^2 / (50000\cdot 10^{-9}) = 32$$

$$= .01\times(.1)^2 / (50000\cdot 10^{-9}) = 2$$

$$= .01\times(.2)^2 / (50000\cdot 10^{-9}) = 8$$

The capacity can be calculated as the sum of three ergodic capacities, one for each SNR.

$$\begin{aligned} C &= \sum B\log_2(1+SNR_i)p(SNR_i) \\ &= 50\times 10^3\left(.2\log_2(1+32)+.5\log_2(1+2)+.3\log_2(1+8)\right) \qquad 27.60 \\ &= 41.4\ kbps \end{aligned}$$

Note here, we calculated a separate capacity for each SNR. We assume no average SNR for the channel.

The equivalent average SNR for an AWGN channel is equal to $0.2\times 32+.5\times 2+.4\times 8=9.8$

The ergodic capacity assuming constant rate for this channel is equal to

$$50\times 10^3\log_2(1+9.8)=51.7\ kbps$$

Compare 51.5 kbps to 41.4 kbps for the fading channel.

Example 7

Find the outage probability of a BPSK signal in a Rayleigh channel with number of antennas equal to 1, 2 and 4. The average SNR per path is equal to 10 dB and the threshold SNR is 7 dB

$$P_{out}=\left[1-e^{-\gamma_0/\bar{\gamma}_i}\right]^M=\left[1-e^{-10\wedge.7/10\wedge 2}\right]^M$$

For M =1, we get, P_{out} = .1466, M = 2, P_{out} = .0215 and for M = 4, P_{out} = 2.13×10^{-7}.

This is reflected in the fact that the smaller the outage probability, smaller the number of transmit antennas that should be used.

Capacity Under a Correlated Channel

We have said a few times already that the MIMO gains come from the independence of the channels. We assume for the development of ergodic capacity that channels created by MIMO are independent. But what happens if there is some correlation among the channels which is what happens in reality due to reflectors located near the base station or the towers. (Usually in cell phone systems, the transmitters (on account on being located high on towers) are less subject to correlation than are the receivers (the cell phones)) We will now examine the effect this has on the system capacity.

The signal correlation, r, between two antennas located a distance d apart, transmitting at the same frequency, is given by zero order Bessel function [5] as

$$r = J_0^2\left(\frac{2\pi d}{\lambda}\right) \qquad 27.61$$

where $J_0(x)$ is the zero-th order Bessel function. Fig. 27.16 shows the correlation coefficient r, plotted between receive antennas vs. d/λ using the Jakes model.

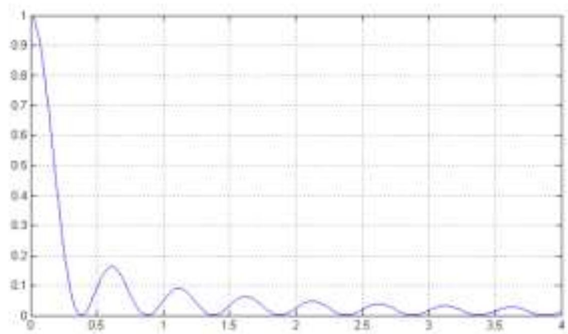


Figure 27.16 – Receive antenna distance d/λ vs. correlation

We see that an antenna that is approximately half a wavelength away experiences only 10% correlation with the first.

To examine the effect that correlation has on system capacity, we replace the channel matrix H in the ergodic capacity equation, $C = \log_2 \det\left(I_{N_R} + \frac{SNR}{N_T}\mathbf{HH}^H\right)$, assuming equal transmit powers, with a correlation matrix, assuming that following normalization holds. (This normalization allows us to use the correlation matrix, rather than covariance.)

$$\sum_{i,j=1}^{N_T, N_R} \left|h_{i,j}\right|^2 = 1 \qquad 27.62$$

Now we write the capacity equation instead as

$$C = \log_2 M \det\left(I + \frac{SNR}{M}\cdot\mathbf{R}\right) \qquad 27.63$$

Where R is the normalized correlation matrix, such that its components $\left|r_{i,j}\right| \le 1$ and

$$r_{ij} = \frac{1}{\sqrt{\sigma_i \sigma_j}} \sum_k h_{ik} h_{jk}^* = \sum_k h_{ik} h_{jk}^* \qquad 27.64$$

We can write the capacity equation as

$$C = \underline{M \cdot \log_2 \det(I + SNR)} + \underline{\underline{\log_2 \det(R)}}$$

The first underlined part of the expression is the capacity of M independent channels and the second is the contribution due to correlation. Since the determinant R is always <= 1, then correlation always results in degradation to the ergodic capacity.

An often used channel model for M = 2, and 4 called the **Kronecker Delta model** takes this concept further by separating the correlation into two parts, one near the transmitter and the other near the receiver, assuming each to be independent of the other. We define two correlation matrices, one for transmit, R_T and one for receiver R_R. The complete channel correlation is assumed to be equal to the Kronecker product of these two smaller matrices.

$$R_{MIMO} = R_R \otimes R_T \qquad 27.65$$

The correlation among the columns of the H matrix represents the correlation between the transmitter and correlation between rows in receivers. We can write these two one-sided matrices as

$$\begin{aligned} \mathbf{R}_R &= \frac{1}{\beta} E\left\{\mathbf{H}\mathbf{H}^H\right\} \\ \mathbf{R}_T &= \frac{1}{\alpha} E\left\{\mathbf{H}^H \mathbf{H}\right\}^T \end{aligned} \qquad 27.66$$

The constant parameters (the correlation coefficients for each side) satisfy the relationship

$$\alpha\beta = Tr\left(R_{MIMO}\right) \qquad 27.67$$

Now if we want to see how correlation at the two ends affects the capacity, we multiply the random channel H matrix with the two correlation matrices as follows.

How do we get these matrices? In some cases, test data is available which can be used, in others, we use a generic form based on Bessel coefficients. If we use the correlation coefficient on each side as a parameter, we can write each correlation matrix as

$$\mathbf{R}_R = \begin{bmatrix} 1 & \sigma & \sigma^2 \\ \sigma & 1 & \sigma \\ \sigma^2 & \sigma & 1 \end{bmatrix} \quad and \quad \mathbf{R}_T = \begin{bmatrix} 1 & \beta & \beta^2 \\ \beta & 1 & \beta \\ \beta^2 & \beta & 1 \end{bmatrix}$$

Now write the correlated channel matrix in a Cholesky form as

$$H = \sqrt{R_T}\, H_w \sqrt{R_R} \qquad 27.68$$

Where H_w is the i.i.d random H matrix, that is now subject to correlation effects.

The correlation at the transmitter is mathematically seen as correlation between the columns of the H matrix and we can write it as R_T. The correlation at the receiver is seen as the correlation between the rows of the H matrix, R_R. Clearly if the columns are similar, then each antenna is seeing a similar channel. When the received amplitudes are similar at each receiver then we are seeing correlation at the receiver.

The H matrix under correlation is ill conditioned, and small changes lead to large changes in the received signal, clearly not a helpful situation.

The capacity of a channel with correlation can be written as

$$C = \log_2 \det\left(I_{N_R} + \frac{SNR}{N_T} R_r^{1/2} H^H R_t^{H1/2} \right) \qquad 27.69$$

When $N_T = N_R$ and SNR is high, this expression can approximated as

$$C = \log_2 \det\left(I_{N_R} + \frac{SNR}{N_T} H_u H_u^{\ H} \right) + \log_2 \det\left(R_r\right) + \log_2 \det\left(R_t\right) \qquad 27.70$$

The last two terms are always negative since $\det(R) \le 0$. That implies that correlation leads to reduction in capacity as shown in the case of a 4×4 system with 20% and 40% correlation.

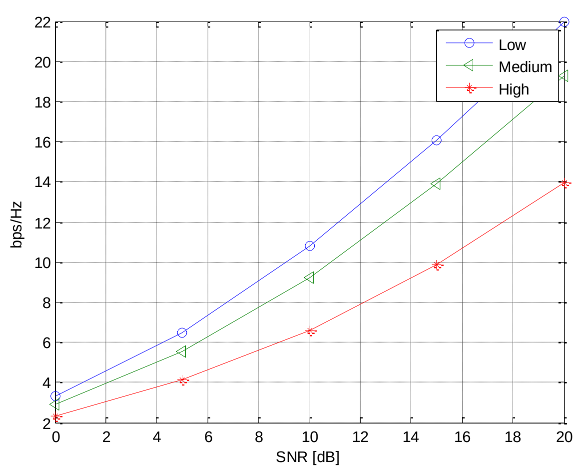


Figure 27.17 – How correlation reduces capacity

Capacity in frequency selective channels

We have assumed that the frequency response is flat for the duration of the single realization of the H matrix. In Figure (27.17) we show a channel that is not flat. Its response is changing with frequency.

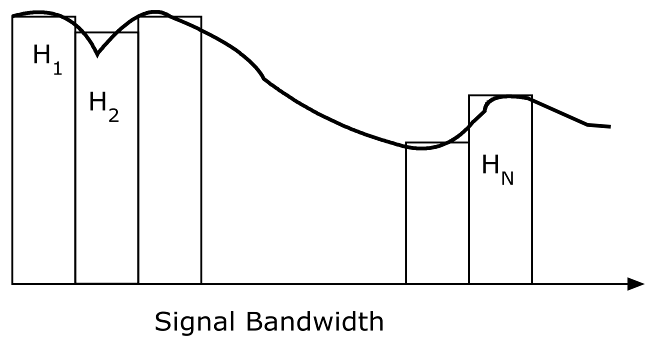


Figure 27.18 - Channel information varies with frequency in a frequency-selective channel

The H matrix now changes within each sub-frequency of the signal. Note that this not time, but frequency. We write the H matrix as a super matrix of sub-matrices for each frequency.

Assume we can characterize the channel in N frequency sub-bands. The H matrix can now be written as $[(N \times N_R),(N \times N_T)]$ matrix. A [3×3] H matrix is subdivided into N frequency and is written as a [18×18] matrix, with [3×3] matrices on the diagonal. The capacity is now calculated same as for a flat channel.

Spatial multiplexing and how it works

We have been assuming that the each of the links in a MIMO system transmit the same information. This is an implicit assumption of obtaining diversity gain. Multicasting provides diversity gain but no data rate improvement. If we could send independent information across the antennas, then there is an opportunity to increase the data rate as well as keep some diversity gain. The data rate improvement in a MIMO system is called **Spatial Multiplexing Gain (SMG)**.

The data rate improvement is related to the number of pairs of the RCV/XMT antennas, and when these numbers are unequal, it is proportional to smaller of the two numbers, N_T, N_R. This easy to see; we can only transmit only as many different symbols as there are transmit antennas. This number is then limited by the number of receive antennas, if the number of receive antennas is less than the number of transmit antennas.

Spatial multiplexing means the ability to transmit higher bit rate when compared to a system where we only get diversity gains because we transmit the same symbol from each transmitter. Just as diversity is defined formally by Equation 27.23, we define spatial multiplexing gain as

$$s = \lim_{SNR \to \infty} \frac{r}{\log(SNR)} \qquad 27.71$$

Where r is data rate that can be obtained as SNR is increased.

Now we ask; should we go for diversity gain or multiplexing gain or maybe a little of both?

Assume that a system has three transmit antennas and five receive antennas, $N_T = 4$ and $N_R = 6$. The diversity order or diversity gain possible is equal to 24, the product of 6 and 4. The SMG however is equal to Min (4, 6) = 4. We cannot achieve both simultaneously. Either we can have diversity gain or multiplexing gain but not both at the same time. We can however, operate in a manner such that we get a diversity gain of 8 and multiplexing gain of 2, as shown by the point marked 8 in the figure below. This figure is called the gain front of the system. It is plotted via the equation below, where diversity gain as a function of the SMG gain is given by [7]

$$27.72 \qquad d(r) = (N_T - r)(N_R - r)$$

Two systems are shown In Figure 27.19. One is for $N_T = N_R = 2$, and the other $N_T = 4$ and $N_R = 6$. The x-axis is diversity order d, or diversity gain and the y-axis is r, the spatial multiplexing gain, SMG. The maximum SMG possible for each system is 2, and 4 respectively. This because the maximum diversity gain possible in a MIMO system is the product of the number of antennas on each side.

The maximum diversity gain possible for these two systems is 4 and 24 respectively.

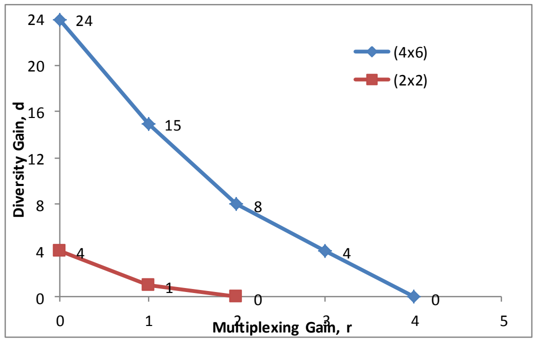


Figure 27.19 – Diversity-multiplexing gain trade-off between Transmit and Receive antennas.

The multiplexing gain is maximum only when diversity gain is 0. When these diversity gains are achieved, no multiplexing gain is possible; hence these values are shown on the y-axis. However, we can use each of these systems in a way that we obtain some combination of diversity gain and multiplexing gain without trying to achieve the maximum of each of these. The design goal is to operate on an optimum front, to obtain a certain diversity gain as well as multiplexing gain. This optimum front is the piece-wise curve shown in Figure 27.19. There are three possibilities for case 1 and 4 for case 2. Which one is optimum? It depends on the system goals.

Space Time Codes

Space Time coding is a field that brings together various techniques for obtaining SMG for a link. There are several techniques that makes it possible to achieve spatial multiplexing gains (SMG), all grouped under the category of Space-Time Coding (STC). The goal of space-time coding is to achieve the maximum possible gain on the optimum gain front based on system goals. Space-Time codes can generally be sub-classified as **Space Time Block Codes (STBC)** and **Space Time Trellis Codes (STTC).** Where Trellis coding is similar to the well-known trellis and convolutional coding of SISO channels, block coding here is different. By block coding we are using space (which means the number of antennas) as one dimension and time as the other. (The book by Hamid Jafarkhani covers this topic well. [11])

Alamouti Code

The first code that defined the space-time block category was discovered by **Siavash Alamouti** and is known famously as the Alamouti Code. This seemingly simple idea is considered one of the most significant advances in MIMO. In fact it was this code that basically set the whole block and trellis coding for MIMO in motion.

We will now consider a Alamouti block code with $N_T = 2$ and $N_R = 1$ or a (2×1) system.
To transmit 2 bits/time, Alamouti got the brilliant idea to transmit two different symbols, s_1, and s_2, one from each antenna. Now we get a multiplexing gain because we are transmitting two symbols in one symbol time, but of course we get no diversity. To make up for that, during the next symbol time, Antenna 1 transmits the negative of the complex conjugate of symbol s_2: $-s_2^*$ and antenna 2 transmits s_1^*, the complex conjugate of symbol one. Of course all we accomplished is that we sent two symbols in two symbol times, no multiplexing gain.

$$s = \begin{bmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{bmatrix} \begin{matrix} \text{Antenna 1} \\ \text{Antenna 2} \end{matrix} \qquad 27.73$$

$$\begin{matrix} t_1 & t_2 \end{matrix}$$

The decoding for the Alamouti (2 × 1) system proceeds as follows: Because there is one receive antenna in this example, the leftmost index of h_{ij} is always 1. Neglecting noise, the received signals r_k, where k is a time index, are

$$\begin{aligned} r_1 &= h_{11}s_1 + h_{12}s_2 \\ r_2 &= -h_{11}s_2^* + h_{12}s_1^* \end{aligned} \qquad 27.74$$

Note that the receiver (but not the transmitter) needs channel state information, namely h_{11} and h_{12}. The receiver multiplies the received waveform by the conjugated weight of that signal. Thus, to form an estimate for s_1, we start by multiplying r_1 by h^*_{11} and r_2 by h^*_{12} yielding

$$\begin{aligned} h_{11}^* r_1 &= |h_{11}|^2 s_1 + h_{11}^* h_{12} s_2 \\ h_{12}^* r_2 &= -h_{11} h_{12}^* s_2^* + |h_{12}|^2 s_1^* \end{aligned} \qquad 27.75$$

Then the estimate for s_1 is found by adding the above Equations (71), and dividing by $|h_{11}|^2 + |h_{12}|^2$ as follows:

$$s_1 = \frac{h_{11}^* r_1 + h_{12} r_2^*}{|h_{11}|^2 + |h_{12}|^2} = s_1 \qquad 27.76$$

We similarly estimate s_2 by multiplying r_1 by h^*_{12} and r_2 by h^*_{11} and so forth, resulting in:

$$s_2 = \frac{h_{12}^* r_1 - h_{11} r_2^*}{|h_{11}|^2 + |h_{12}|^2} = s_2 \qquad 27.77$$

Of course, you see from Equation 27.77 that to estimate or recover the transmitted symbols, the receiver needs to know the channel coefficients. You should also note that so far this scheme has not provided any data rate or multiplexing gain, only diversity. That’s because we sent just two symbols in two time periods.

But we do get diversity gains that are substantial, as shown in Figure 27.20, with the Alamouti scheme using one receive antenna yields a gain of about 8.5 dB over the corresponding SISO channel at a $P_B = 2 \times 10^{-3}$. If a second receive antenna is added, the code achieves an additional 3.5 dB gain at the same P_B. This simple scheme is very popular because it can be introduced to existing systems for providing link-quality improvements without any major system modifications. It is part of the W-CDMA and CDMA-2000 standards.

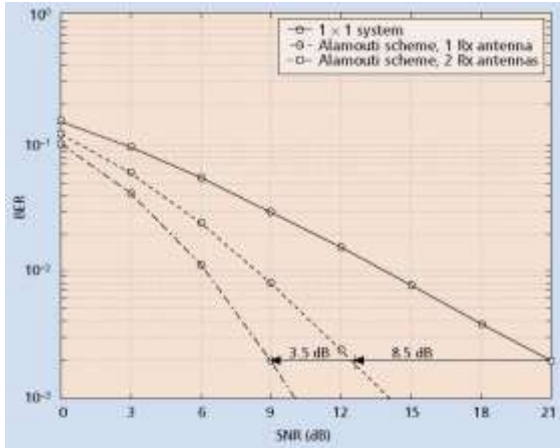


Figure 27.20 - BER performance of the Alamouti scheme in flat fading. Source: J. Mietzner and P. A. Hoeher, "Boosting the performance of wireless communication systems: theory and practice of multiple-antenna techniques," *IEEE Communications Magazine*, October 2004, pp. 40-46.

The key diversity-creating feature in the Alamouti scheme is the orthogonality between sequences generated by the two transmit antennas.

The code's success has led to a wave of generalized developments for an arbitrary number of transmit antennas. Such a generalized STBC is defined by a $(N_T \times p)$ matrix **C** whose entries are transmission symbols (possibly encoded with other codes, and possibly complex). The columns p of the matrix represents time slots, and the rows (designed to be orthogonal) represent transmit antennas.

Equation (27.78) depicts such a **C** matrix for $N_T = 4$. At time 1, the first column of four code symbols are transmitted from antennas 1-4, respectively. At each successive time, the next column is sent from antennas 1-4 respectively, and so forth. Space-time codes can provide a maximum diversity less than or equal to $N_T \times N_R$. Thus, for $N_R = 1$, the encoder provides a diversity of 4 (maximum possible with 4 transmit antennas and 1 receive antenna). For this code, there are 4 symbols sent during each block of 8 time slots. We see in Equation (27.78), a rate r = ½ code. The scheme provides a 3-dB received power gain that stems from 8 slots used to send 4 symbols. This compensates for the rate loss.

$$C = \begin{bmatrix} c_1 & -c_2 & -c_3 & -c_4 & c_1^* & -c_2^* & c_3^* & -c_4^* \\ c_2 & c_1 & c_4 & -c_3 & c_2^* & c_1^* & c_4^* & -c_3^* \\ c_3 & -c_4 & c_1 & c_2 & c_3^* & -c_4^* & c_1^* & c_2^* \\ c_4 & c_3 & -c_2 & c_1 & c_4^* & c_3^* & -c_2^* & c_1^* \end{bmatrix} \qquad 27.78$$

Such codes allow for simple maximum likelihood decoding [14], and provide the maximum diversity that can be obtained for a given number of transmit and receive antennas. For $N_T > 2$, it is not known whether codes exists that have a rate 1.

Space-Time Trellis Codes

Just like space-time block codes (STBC), **Space-Time Trellis Codes** (STTC), can provide a diversity benefit equal to the number of transmit antennas. In addition, without any loss in bandwidth efficiency, a STTC can also provide a coding gain that depends on the complexity of the code (number of states in the trellis).

This Figure below shows a 4-state trellis for encoding QPSK symbols.

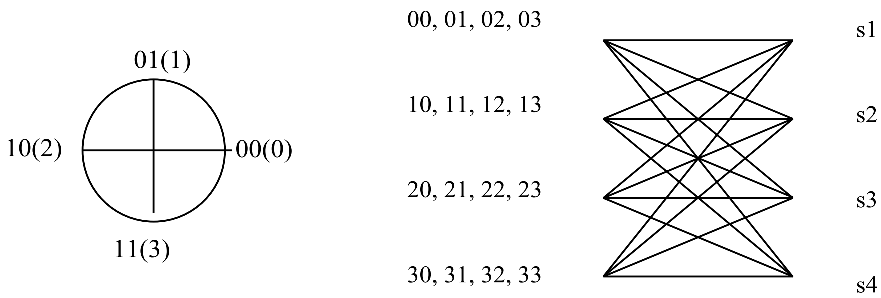


Figure 27.22 – Trellis coding for a (2×2) QPSK MIMO system

The constellation on the right shows the four QPSK symbols (numbered 0 to 3, represented by 1, j, -1, -j) and their bit assignments. On the right we have a 4-state trellis. We are assuming that we will be using two antennas. We know this because at each state we have four groups of symbols, of two symbols each. Each group stands for symbols we will transmit over each antenna for that state. At state 1, we have 00, 01, 02, 03. These numbers stand for the symbol number in the constellation.

Let’s say we want to transmit a bit sequence, 10 00 11 10. This maps to symbols: 2, 0, 3, 2. We always start in state 1, so we are at the top left side.

State	Incoming	TX 1	TX 2
1	2	2	0
3	0	0	2
1	3	3	1
3	2	2	3

The first input symbol is 2, which is the third symbol of the constellation. We pick the third group at state 1, corresponding to the fact that third symbol is to be sent, which is 02. The first antenna will transmit the first symbol in this group which is 2 and the second antenna will transmit the symbol 0 from this third group in the row. Note how we mapped the incoming symbol to two new symbols, one for each antenna. Since we are in the third group of the four, we jump to state 3 ready to map the next incoming symbol.

The next symbol to transmit is 0. We pick the first group in state s_3 -because 0 is the first symbol in the set- which is 20. Antenna 1 transmits symbol 0 and antenna 2 will symbol 2. Now we jump to state 1, because that is where we were. The incoming symbol is 3. The corresponding group is 03. Antenna 1 transmits symbol 3 and antenna 2 transmits 0. This is shown in table below.

Remember that in trellis encoding, we always start and end in state 1. The number of states is a function of the constraint length of the code and not modulation. It is related to the coding gain that the trellis can provide.

Below we show a 8-state trellis code using 8PSK modulation.

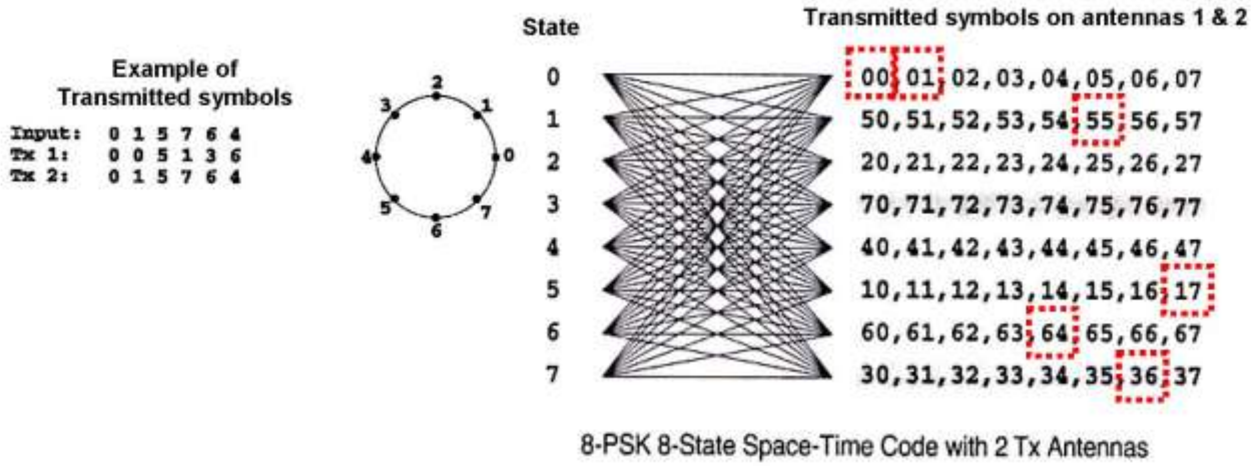


Figure 27.23 - Example of 8-state 8-PSK space-time trellis code (2 Tx antennas). Source: V. Tarokh, et. al., "Space-time codes for high data rate wireless communications," *IEEE Trans. Info. Th.*, March 1998.

There is one other category of space time codes, called **Layered Space Time codes**, invented by Gerald Fochini. We will not discuss those here and are left for your study.

Multi-User MIMO

There is a natural mapping of MIMO to the way cellular systems work. A base station transmitting to multiple users can collectively be thought of as a matrix channel transmitted to multiple users. Similarly the users transmitting to and through the base station can be thought of as a matrix multiple-access channel. The base station can use either a single antenna or many, creating a SIMO or a MIMO channel. We can think of the mobile users, even though they may be transmitting on just one antenna, as creating independent channels in a MIMO system. For K mobiles, we would have K uplink transmit antennas albeit they are not co-located nor are they necessarily uncorrelated, and a similar number of receive antennas at the base station to receive these independent signals. We then have a (K×K) multi-user, or MIMO-MU system. If the receivers have more than one receive or transmit antennas, say N, then we have a (K×N, K) MIMO-MU system. For a system of 8 multiple users, sharing the same MIMO channel, with 2 antennas each, we would have a (16, 8) MIMO-MU system, that's 16 uplink channels to each receive antenna on the base.

In a multiuser MIMO system, the channel from the base station to all the users, called the downlink channel is referred to as the Broadcast Channel (BC). The base station communicates with all users at the same frequency using. This form of MIMO is called MIMO-MU for multiple users as opposed to MIMO-SU which we have been discussing so far. Frequency reuse in satellite communication is a form of MIMO-MU. Here the satellite creates multiple beams at same frequencies to communicate with users that are not spatially co-located, such as beams pointed to different cities. This method is also called Space Division Multiple Access (SDMA) but can also be considered a MIMO-MU. In cellular systems MIMO-MU allows users in one cell - spatially separated from each other- to communicate with a base station via vector/matrix MIMO channel. In the downlink scenario, the receivers are independent so it is clear we cannot use typical MIMO receiver strategies such as MRC.

Here we summarize some differences and advantage of the MIMO-MU vs. the MIMO-SU as listed in [12] MIMO-SU is a point to point link with a defined capacity, where MIMO-MU has no defined capacity but is characterized instead by capacity regions.

In MIMO-MU each user has a capacity that is designed to be approximately equal. The channel is considered to be in outage if even one of these channels suffers outage. In MIMO-SU since there are multiple paths for each user, even though one path may have outage, the channel may still be operational because it has path diversity.

In MIMO-MU the users are geographically distributed, so the near-far problem of power management still exists. The base station may not always be able to manage these power differences among the users. Water-filling algorithm cannot be used because a minimum data rate is required on all paths, so allocating power to the weak channel penalizes others.

Only uplink space-time coding can be done cooperatively. The users can not cooperate in decoding.

In MIMO-SU, the uplink and downlink capacities are equal if channel is known. In MIMO-MU this aspect is still under research and development. In MIMO-MU if the channel is not knows to transmitter, the channel penalty is much greater than MIMO-SU where lack of CSIT does not have a big impact on channel capacity.

The forward channel in MIMO-MU is referred to as MIMO Broadcast Channel, or MIMO-BC. The reverse channel is called MIMO Multiple Access Channel, or MIMO-MAC. We talk briefly about each of these and their design issues.

MIMO-MAC

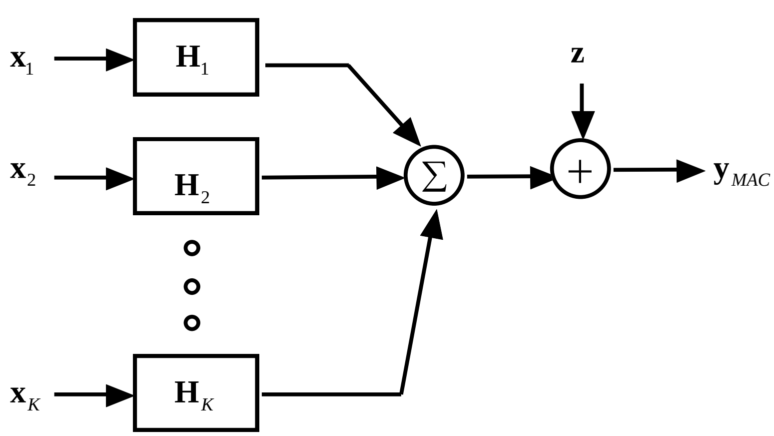


Figure 27.24 – K independent users in multi-user Uplink MIMO

Consider a MIMO-MAC system with N_T antennas at the base station and K independent users. Let's assume that each user has just one antenna. The energy per user is not constant nor is it assumed to be normalized as we do in MIMO-SU. However, we assume that the users do apply power-control to manage their own transmit power as is typical in a cellular system. The received signal at the base station is given as

$$\begin{aligned} \mathbf{y} &= \sum_{i=1}^{K} \mathbf{h}_i \mathbf{x}_i + \mathbf{z} \\ &= \mathbf{Hx} + \mathbf{z} \end{aligned} \qquad 27.79$$

The covariance matrix of independent transmitted symbols $\mathbf{x}$, $\mathbf{R}_{xx} = E\left\{\mathbf{xx}^H\right\}$ is a diagonal matrix of

$$R_{ss} = diag\left\{E_{s,1}, E_{s,1} \cdots, E_{s,K}\right\} \qquad 27.80$$

The capacity of a MIMO-MU is described as a capacity region within which each user can obtain a data rate based on a tradeoff with other users. The problem with MU systems is that all decoding (at the receiver) are independent and no coordination is possible. Hence we can see that a MU system would not be able to obtain the same capacity as a single user system.

We will assume that there are just 2 users in order to show the concept of the capacity region and decoding. For more than two users, the optimum capacity region becomes the surfaces of a polyhedral. Figure 27.25 shows the shape of the capacity region for the case of a MIMO-MU.

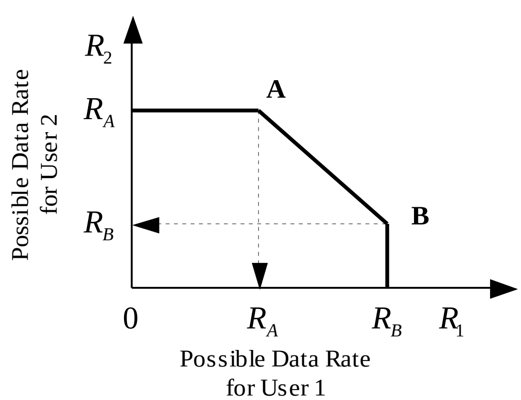


Figure 27.25 Operating front

The capacity region has been shown to satisfy [Surad,1998] such that the data rate (which is the capacity) possible for each user obeys the follows constraints

$$R_1 \leq \log_2 \det\left(I_{N_B} + \frac{E_{s,1}}{N_0} \|h_1\|^2 \right) \qquad 27.81$$

$$R_2 \leq \log_2 \det\left(I_{N_B} + \frac{E_{s,2}}{N_0} \left\| h_2 \right\|^2 \right) \quad 27.82$$

$$R_1 + R_2 \leq \log_2 \det\left(I_2 + \frac{E_{s,1}}{N_0} h_1 h_1^H + \frac{E_{s,2}}{N_0} h_2 h_2^H \right) \quad 27.83$$

In Figure 27.25 the capacity region is shown. The optimum data rates are along the line AB. However in combination on the edges or inside the surface is also achievable if joint decoding can be used.

MIMO-BC

MIMO-BC is the link from the base station to all the mobiles. This channel can be thought of as a MIMO channel but is more complex than the single user case, because the receivers are completely independent and cannot take advantage of the knowledge of the other paths. The capacity region for the MIMO-BC is considered an unsolved problem. A result was described as “writing on dirty paper” by Costa.

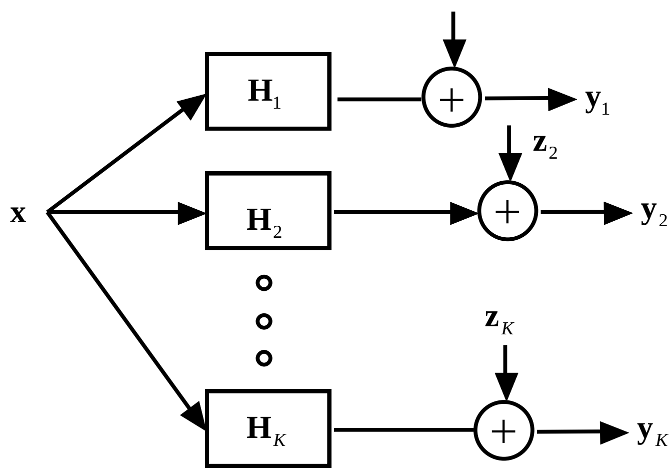


Figure 27.26 MIMO-BC

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Bibliographic Notes

In the writing of this paper, I relied heavily on some books and papers. The three books I often found myself thumbing through, in order are: Wireless Communications by Andrea Goldsmith, *Fundamentals of Wireless Communications* by D. Tse and P. Viswanath and Digital Communications by Proakis, 5th Edition. The examples used in this paper are inspired by the Andrea Goldsmith book, the one I consider the best. She has a lot of examples in her book which truly help with understanding.

The other three books that I also found helpful were: Digital Communications by Barry, Less and Messerschmitt, Introduction to Space-Time Wireless Communications by Paulraj, Nabar and Gore and *MIMO-OFDM Wireless Communications with Matlab book by* Yong Soo-Cho, and Won Young Yan. I used the Matlab code in this book to create some of the capacity graphs.

With papers, the one I read several times was by Gesbert's [13] and the one by Foschini. [1] I read many others but these two stand out. Agilent also has an excellent tutorial online. And of course the David Tse video lecture is fantastic. IEEE also has an audio lecture that is excellent.

It took me a while to get all my ideas together and I read many papers and checked nearly all the books written on the topic. While writing about these, I often could not figure out who the original source was. This is not good as I do want to credit to whom it is is due. If you the reader feel that I have not properly credited you or someone else in this paper, please do let me know.

In writing, Bernard wrote the first draft and then I added and subtracted from it. The paper is now nearly 50 pages but it is still lacking in many areas. I was not able to cover STC decoding, nor the code performance issues. The section on multi-user is also short. But I hope that what is here will help illuminate the topic and get you started.

For an up-to-date copy of this tutorial, please check www.complextoral.com