Л. В. Тарасов
А. Н. Тарасова

Беседы о преломлении света

Под редакцией В. А. Фабриканта

Издательство «Наука». Москва

L. V. Tarasov
A. N. Tarasova

Discussions on refraction of light

Edited by V. A. Fabrikant

Translated from the Russian
by Vladimir Kolesnikov
and Nicolas Kulanin

Mir
Publishers
Moscow

Л. В
А. Н

Бес

Под

Изда

First published 1984

На английском языке

Preface

Why does a beam of light change its direction when passing through the interface between two media? Why does the setting sun appear oblate on the horizon? What causes mirages? Why does a prism disperse sunlight into different colours? How can one calculate the angular dimensions of a rainbow? Why do distant objects appear close when we view them through a telescope? What is the structure of the human eye? Why does a light ray get broken into two in a crystal? Can the plane of the polarization of a ray be turned? Can light rays be bent at will? Is the refractive index controllable?

This book will give the reader answers to all these questions. He will get to know how the law of refraction was discovered, how Newton's theory of the refraction of light in the atmosphere was nearly lost forever, how Newton's experiments changed radically the old ideas concerning the origin of colours, how the telescope was invented, how it took twenty centuries to understand the anatomy of human vision, and how difficult it was to discover the polarization of light.

To make the historical and the physical aspects of the book more convincing, the authors have introduced a number of problems and their detailed solutions, geometrical constructions, and optical diagrams of some instruments and devices. No doubt, the reader will get a better understanding of some excerpts from the classics of physical optics (for example, Newton's "Optics" or Huygens' "Treatise on Light") after they have been illustrated with the help of diagrams, constructions and concrete problems.

Thus, as he explores the world of refracted rays, the reader will be able to familiarize himself not only with the physics of the topics being considered but also with the evolution of some of the concepts of physics and their practical applications to problems, constructions and optical schemes. It is the authors' hope that this journey will be both instructive and enjoyable.

The authors are greatly obliged to Professor V. A. Fabrikant for his editing and for the many valuable suggestions he made.

L. Tarasov
A. Tarasova

Contents

Chapter One

Light rays at the interface between two media

A Ring at the Bottom of a Water-Filled Vessel. Take a shallow vessel with opaque walls; a mug, a tin or a pan will be suitable. Place a ring at the bottom of the vessel and look at it at an angle so that you can see a part of the bottom

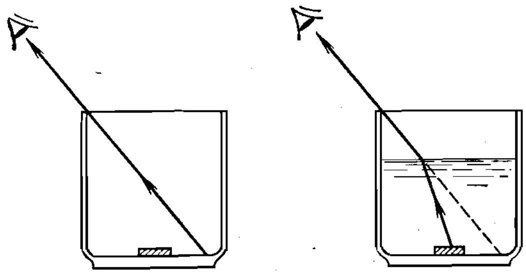

Fig. 1.1.

without seeing the ring. Ask somebody to fill the vessel with water without moving it. When the level of the water has reached a certain height, you will see the ring lying at the bottom. This unsophisticated experiment is an invariable success. It illustrates in a spectacular way the *refraction* of light rays at the interface between water and air (Fig. 1.1).

The experiment described above has been known for a long time. In 1557 a translation of Euclid's "Catoptrics" (3rd century B. C.) was published in Paris. It contains the following statement: "If an object is placed at the bottom of a vessel so that the object cannot be seen, it will come back into view if the vessel is filled with water, the distance remaining unchanged". True, the experiment described has no direct bearing on the question dealt with in Euclid's book. The latter is devoted to catoptrics, which was at that time the name of the branch of optics referring to the reflection of light, whereas the refraction of light was studied by dioptrics. The experiment with a ring at the bottom of a vessel is commonly supposed to have been added by the translator of the book. But still, there is not a shade of doubt that the experiment is about twenty centuries old. It is described in other ancient sources, particularly, in Cleomedes' book (50 A. D.) "The Circular Theory of the Heavenly Bodies". Cleomedes wrote: "Is it not possible that a light ray passing through humid layers of air should curve...? This would be similar to the experiment with a ring placed at the bottom of a vessel, which cannot be seen in an empty vessel, but becomes visible after the vessel is filled with water."

Consider quite a modern problem using the ancient experiment. *In a cylindrical vessel whose height equals the diameter of its bottom, there is a disc in the centre of the bottom whose diameter is half that of the bottom of the vessel. The observer can just see the edge of the bottom (naturally, he cannot see the disc lying at the bottom). How much of the vessel's volume has to be filled with water*

so that the observer can just see the edge of the disc? The refractive index of water $n = 4/3$.

Designate the diameter of the bottom of the vessel as D, and the level of the water in the vessel at which the observer can see the edge of the disc as H (Fig. 1.2).

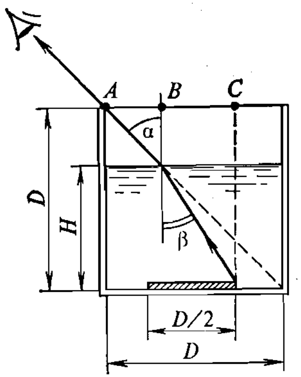

Fig. 1.2.

The law of the refraction of light rays is described by the relation

$$\frac{\sin\alpha}{\sin\beta} = n, \tag{1.1}$$

Rewrite the equation $AB + BC = AC$ as $(D - H)\tan\alpha + H\tan\beta = 3D/4$ or (bearing in mind that $\tan\alpha = 1$ under the conditions of the problem)

$$\frac{D}{H} = 4(1 - \tan\beta). \tag{1.2}$$

Passing from $\tan\beta$ to $\sin\beta$ and using Eq. (1.1), we have

$$\tan\beta = \frac{\sin\beta}{\sqrt{1-\sin^2\beta}} = \frac{\sin\alpha}{\sqrt{n^2-\sin^2\alpha}} = \frac{1}{\sqrt{2n^2-1}}. \tag{1.3}$$

Substituting (1.3) into (1.2) we find

$$\frac{D}{H}=4\left(1-\frac{1}{\sqrt{2n^2-1}}\right).$$

Since $n = 4/3$, $H/D = 0.67$. Thus, the observer will be able to see the edge of the disc when water fills 0.67 of the vessel's volume.

Ptolemy's Experiments. In the problem considered above the *law of refraction* (1.1) was used. Many investigations conducted over a

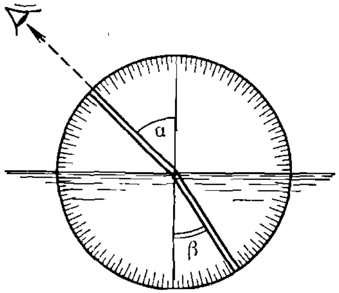

Fig. 1.3.

long period of time preceded the discovery of this law. They date back to the 2nd century A. D., when Ptolemy tried experimentally to determine the relationship between the angles which the incident and the refracted rays make with the normal to the interface between media.

Ptolemy used a disc graduated in degrees. The ends of two rulers were attached to the centre of the disc, so that the rulers could be turned about the fixed axis. The disc was half-submerged in water (Fig. 1.3), and the rulers were positioned

in such a way that they both seemed to be in a straight line when viewed from the top. Ptolemy fixed the upper ruler in different positions (corresponding to different values of the angle α) and experimentally found the corresponding position of the lower ruler (the corresponding value of the angle β). It followed from Ptolemy's experiments that the ratio $\sin \alpha/\sin \beta$ laid within the range from 1.25 to 1.34, i.e. it was not quite constant. Thus, Ptolemy failed to discover the exact law of the refraction of light.

The Discovery of the Law of Refraction by Snell. Over four centuries passed before the law of refraction was at last established. In 1626 the Dutch mathematician Snell died. Amidst his papers a work was found, in which, in fact, he was found to have formulated the law of refraction. To illustrate Snell's conclusions, turn to Fig. 1.4. Assume that FO is the interface between the media; the rays are incident on the interface at point O. The figure shows three rays (*1*, *2*, and *3*); α_1, α_2, and α_3 are their angles of incidence, and β_1, β_2, and β_3 are the angles of refraction. Erect the perpendicular FG at a point F chosen at random on the interface between the media. Designate the points at which the refracted rays *1*, *2*, and *3* cut the perpendicular as A_1, A_2, and A_3, and those at which the extensions of the incident rays *1*, *2*, and *3* cut it (in the figure the extensions are represented by dashed lines) as B_1, B_2, and B_3. By experiment Snell established that

$$\frac{OA_1}{OB_1}=\frac{OA_2}{OB_2}=\frac{OA_3}{OB_3}.$$

Thus, the ratio of the length of the refracted ray from the point O to where it crosses FG to the length of the extension of the incident ray from O

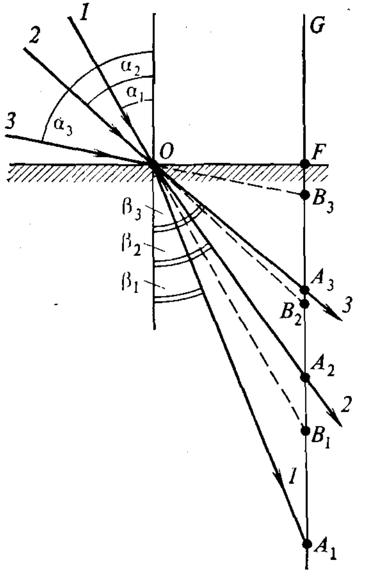

Fig. 1.4.

to where it crosses FG is constant for every ray incident on the interface:

$$\frac{OA_i}{OB_i} = \text{const} \tag{1.4}$$

(the index i indicates different rays).

The commonly accepted formula for the law of refraction follows immediately from (1.4).

Since $OA_i \sin\beta_i = FO$ and $OB_i \sin\alpha_i = FO$, formula (1.4) gives

$$\frac{\sin\alpha_i}{\sin\beta_i} = \text{const.} \tag{1.5}$$

Thus, the ratio of the sine of the angle of incidence to the sine of the angle of refraction is *constant* for a given pair of substances.

Descartes' Interpretation of the Law of Refraction. Descartes' Error. However, for some unknown reason Snell did not publish his work. The first publication which contains the wording of the law of refraction does not belong to Snell but to the famous French scientist René Descartes (1596-1650).

Descartes was interested in physics, mathematics and philosophy. He had an original and, undoubtedly, vivid personality, and opinions about him were many and controversial. Some of Descartes' contemporaries accused him of making use of Snell's unpublished work on the refraction of light. Whether Descartes did or did not see Snell's work, the accusation is groundless. The fact is that Descartes formulated the law of refraction on the basis of his own ideas about the properties of light. He deduced the law of refraction from the assumption that light travels at different velocities in different media, i.e. his law was arrived at *theoretically*.

Curiously enough, Descartes formulated the law of refraction using the erroneous assumption that the velocity of light increases when it goes from air into a denser medium. Today, we find Descartes' ideas about the nature of light rather confused and naive. He regarded the propagation of light as the transferrence of pressure through ether, a substance which, it was supposed, surrounded and penetrated everything. His work entitled "Dioptrics" reads: "Since there is no vacuum in nature and since each body has pores in it, it is necessary that these pores be

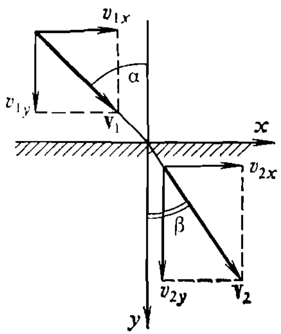

Fig. 1.5.

filled with matter, that is rather very rarefied and fluid, and which propagates incessantly from celestial luminaries towards us.... Light is nothing but a kind of motion or effect produced in the rather rarefied matter filling the pores of the bodies." When analysing the refraction of light, Descartes used an analogy with a ball thrown into water. He claimed that "light rays conform to the same laws as the ball".

Descartes' ideas regarding the refraction of light can be illustrated by Fig. 1.5. Assume that v_1 is the velocity at which light pressure is transferred in the first medium, and v_2 is the velocity in the second medium. Descartes resolved both

vectors into two components—one parallel to the media interface (the x-component) and one perpendicular to the interface (the y-component). He supposed that when light leaves one medium and enters the other it is only the y-component of $\mathbf{v}$ that changes, and in a denser medium this component is greater. Putting it differently, we get

$$v_{1x} = v_{2x}; \quad v_{1y} < v_{2y}. \tag{1.6}$$

The figure shows that

$$\frac{\sin\alpha}{\sin\beta} = \frac{v_{1x}/v_1}{v_{2x}/v_2} = \frac{v_2}{v_1}. \tag{1.7}$$

Descartes' major error was that he supposed that light propagates faster in a denser medium, whereas in reality it is *the other way around.* "The harder the particles of a transparent body", was Descartes' rather obscure reasoning, "the easier they let light pass through, for the light does not need to push any particles out of their place in the way a ball pushes aside particles of water to make its way through...".

Descartes' error was put right by Huygens and Fermat.

Huygens's Principle. The famous Dutch physicist and mathematician Christiaan Huygens (1629-1695) considered the propagation of light to be a wave process. Huygens supposed that light was in fact constituted by waves propagating through ether.

He looked upon the propagation of light waves in the following manner. Assume that the light wave is plane, the cross section of its wavefront being a straight line. Let it be line AA in

Fig. 1.6. Light reaches every point of AA simultaneously and, according to Huygens, all these points start functioning simultaneously as point sources of secondary spherical waves. As Huygens stressed in his "Treatise on Light", "...light

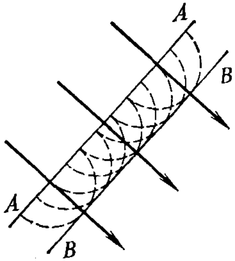

Fig. 1.6.

propagates in consecutive spherical waves". After a certain period of time Δt, these wave-fronts will create the situation shown in Fig. 1.6 by dashed semicircles. Draw the envelope of the fronts, which is actually the line BB. It corresponds to the new position of the plane wave-front. It can be said that within the time Δt the front of the light wave has moved from AA to BB. Naturally, every point on BB can also be regarded as the source of secondary light waves. In the figure, light rays are represented by arrows. At every point in space a light ray is *perpendicular* to the wave-front passing through the point.

This method of representing consecutive position of the wave-front became known as Huygens' method. It is also referred to as *Huygens' principle* and is formulated as follows: *every point reached by a light disturbance becomes in its turn the source of secondary waves, the surface enveloping these secondary waves at a given instant indicates the position of the actual propagating wave-front.*

Huygens' Principle and the Law of Refraction. Huygens deduced the law of refraction of light

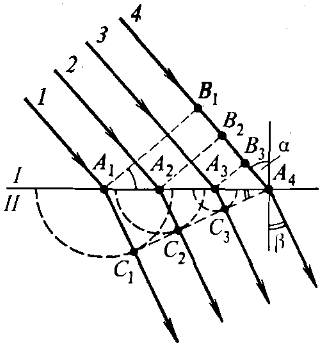

Fig. 1 7.

using his principle (Fig. 1.7). Assume that a plane light wave is incident at an angle α on a surface A_1A_4, which is the interface between two media, for example, water and air. Let the velocity of light in the first medium (air) be v_1, and the velocity in the second medium (water) be v_2. According to Huygens' correct reasoning, $v_1 > v_2$. Four light rays are shown by arrows in the figure; the line A_1B_1 (dotted) shows where the wave-front is at the moment when the ray *1* reaches the interface between the media. According to Huygens, at the same moment, the point A_1 becomes the source of a secondary spherical wave. Note that this wave continues to propagate in both the first and second media, generating reflected and refracted bundles of rays, respectively. We shall confine ourselves to the refracted waves. The dashed semicircle with its centre at A_1 shows the front of the spherical wave under consideration after a period of time Δt_1 during which the ray *4* travels from B_1 to A_4. We can write that

$$\Delta t_1 = \frac{B_1A_4}{v_1} = \frac{A_1C_1}{v_2}. \quad (1.8)$$

When the ray *2* reaches the interface, A_2 becomes the source of a secondary wave. The semicircle with the centre at A_2 (dotted) represents the front of this wave after a certain period of time Δt_2, during which the ray 4 travels from B_2 to A_4. Hence $\Delta t_2 = B_2A_4/v_1 = A_2C_2/v_2$. When the interface is reached by a ray *3*, point A_3 becomes the source of a secondary wave. The dotted semicircle with its centre at A_3 is actually the front of this wave after Δt_3, during which the ray *4* travels from B_3 to A_4, hence $\Delta t_3 = B_3A_4/v_1 = A_3C_3/v_2$. The line C_1A_4 is the envelope of the semicircles shown in the figure; it corresponds to the wave-front of the refracted bundle of rays at the moment the ray *4* reaches the interface. It is clear from the figure that $\sin\alpha = \frac{B_1A_4}{A_1A_4}$, $\sin\beta = \frac{A_1C_1}{A_1A_4}$, and therefore $\frac{\sin\alpha}{\sin\beta} = \frac{B_1A_4}{A_1C_1}$. Using (1.8), we have

$$\frac{\sin\alpha}{\sin\beta} = \frac{v_1}{v_2}. \quad (1.9)$$

Unlike (1.7), the correct relation between the velocities is written here.

In this way the constant relation $\sin\alpha/\sin\beta$ discovered by Snell was explained from two opposing theoretical premises: Descartes' erroneous assumption that the velocity of light in a dense medium is greater than it is in air and the correct though opposite assumption made by Huygens. You can thus see how one experiment can be used to substantiate different theories. It stands to reason that a theory is always based on and checked against an experiment. However, one should refrain from putting forward a new theory if it is based upon *insufficient* number of experiments. The history of physics has records of other examples, apart from Descartes' error, when theories formulated on the basis of insufficient experimental data were later proved to be incorrect by further tests. The creation of a new theory calls for a well-considered system of experiments to check it for viability as well as its compliance with other known facts and theories. A brilliant example here is the system of experiments with prisms the great Isaac Newton carried out. He used them to create his famous theory of the origin of colour. This will be given special consideration later (in Chapter Five), but now we should go back to the law of refraction.

We introduce the *refractive index n* for the given medium. According to the present-day views

$$n = \frac{c}{v}, \quad (1.10)$$

where c is the velocity of light in vacuum (this fundamental physical constant equals $2.9979 \times \times 10^8$ m/s), and v is the velocity of light in the medium under consideration. Using (1.10) and (1.9), we can rewrite the law of refraction as follows:

$$\frac{\sin\alpha}{\sin\beta}=\frac{n_2}{n_1}, \tag{1.11}$$

where n_1 and n_2 are the refractive indices of the first and the second media, respectively. If light passes from air to a denser medium, for example, water or glass, the velocity of light in air can be assumed to be equal to c, i.e. the refractive index of air is unity. Then, we can write

$$\frac{\sin\alpha}{\sin\beta}=n, \tag{1.12}$$

where n is the refractive index of the denser medium.

Fermat's Principle (the Principle of Least Time). However, let us go back to the 17th century to familiarize ourselves with the investigation of Pierre Fermat (1601-1665), a well-known French mathematician. Fermat became interested in the refraction of light some years before Huygens. He came up with a general principle concerning the way light rays travel in different circumstances and, in particular, when light rays pass through an interface between two media. This is known as *Fermat's principle* or the *principle of least time*. The wording of the principle is: *the actual path of the propagation of light (the trajectory of a light ray) is the path which can be covered by light within the least time in comparison with all other hypothetical paths between the same points.*

Evidently, Fermat first conceived his idea when considering the statement of Hero of Alexandria (2nd century B. C.) that reflected light travels from one point to another along the shortest path. True, it is clear from Fig. 1.8 that ABC which complies with the law of reflection is shorter than any other imaginable path from the point A to C, for example, the path ADC. The length of ABC equals the length of the line AC_1, whereas the length of ADC actually equals the length of the broken line ADC_1 (C_1 is the mirror image of the point C).

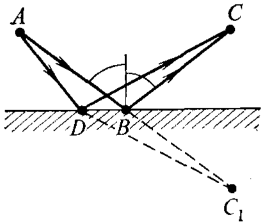

Fig. 1.8.

It is quite obvious that the refraction of light does not obey the principle of the shortest path. Taking this fact into consideration, Fermat suggested that the *principle of shortest path* be replaced with the *principle of least time*. Fermat's principle explains the reflection of light in a very clear way. Besides, unlike the principle of shortest path, it accounts for the refraction of light as well.

The well-known "Feynman Lectures on Physics" have the following passage: "To illustrate that the best thing to do is not just to go in a straight line, let us imagine that a beautiful girl has fallen out of a boat, and she is screaming for help in the water at point B. The line

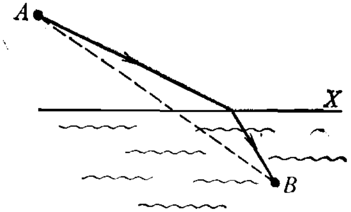

Fig. 1.9.

marked X is the shoreline (Fig. 1.9). We are at point A on land, and we see the accident, and we can run and can also swim. What do we do? Do we go in a straight line?... By using a little intelligence we would realize that it would be advantageous to travel a little greater distance on land in order to decrease the distance in the water, because we go much slower in the water."

Deduction of the Law of Refraction from Fermat's Principle. Now let us reason absolutely rigorously. Let the plane S be the interface between medium *1* and medium *2* with the refractive indices $n_1 = c/v_1$ and $n_2 = c/v_2$ (Fig. 1.10*a*). Assume, as usual, that $n_1 < n_2$. Two points are given—one above the plane S (point A), the other under the plane S (point B). The various distances are: $AA_1 = h_1$, $BB_1 = h_2$, $A_1B_1 = l$. We must find the path from A to B which can be covered by light faster than it can cover any other hypothetical path. Clearly, this path must consist of two straight lines, viz.

AO in medium *1* and OB in medium *2*; the point O in the plane S has to be found.

First of all, it follows from Fermat's principle that the point O must lie on the intersection of S and a plane P, which is perpendicular to S and passes through A and B.

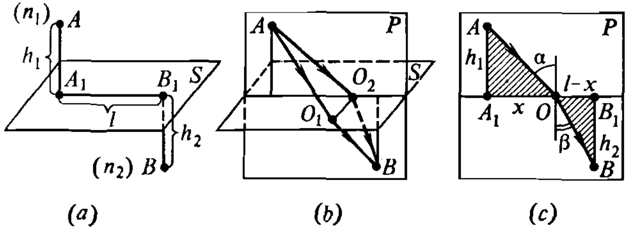

Fig. 1.10.

Indeed, let us assume that this point does not lie in the plane P; let this be point O_1 in Fig. 1.10*b*. Drop the perpendicular O_1O_2 from O_1 onto P. Since $AO_2 < AO_1$ and $BO_2 < BO_1$, it is clear that the time required to traverse AO_2B is less than that needed to cover the path AO_1B. Thus, using Fermat's principle, we see that the first law of refraction is observed: the incident and the refracted rays lie in the same plane as the perpendicular to the interface at the point where the ray is refracted. This plane is the plane P in Fig. 1.10*b*; it is called the plane of incidence.

Now let us consider light rays in the plane of incidence (Fig. 1.10*c*). Designate A_1O as x and $OB_1 = l - x$. The time it takes a ray to travel from A to O and then from O to B is

$$T = \frac{AO}{v_1} + \frac{OB}{v_2} = \frac{\sqrt{h_1^2 + x^2}}{v_1} + \frac{\sqrt{h_2^2 + (l-x)^2}}{v_2}. \qquad (1.13)$$

The time depends on the value of x. According to Fermat's principle, the value of x must minimize the time T. Those familiar with basic mathematical analysis know that at this value of x the derivative dT/dx equals zero:

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{h_1^2 + x^2}} - \frac{l-x}{v_2\sqrt{h_2^2 + (l-x)^2}} = 0. \qquad (1.14)$$

Now,

$$\frac{x}{\sqrt{h_1^2+x^2}}=\sin\alpha, \quad \text{and} \quad \frac{l-x}{\sqrt{h_2^2+(l-x)^2}}=\sin\beta,$$

consequently,

$$\frac{\sin\alpha}{v_1}-\frac{\sin\beta}{v_2}=0. \tag{1.15}$$

The second law of refraction described by the ratio (1.9) immediately follows from (1.15).

True, Fermat himself could not use (1.14) as mathematical analysis was developed later by Newton and Leibniz. To deduce the law of the refraction of light, Fermat used his own maximum and minimum method of calculus, which, in fact, corresponded to the subsequently developed method of finding the minimum (maximum) of a function by differentiating it and equating the derivative to zero.

Application of Fermat's Principle. Fermat's principle can be illustrated by the following

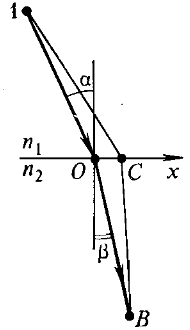

Fig. 1.11.

example. Let a light ray travel from A to B by passing through an interface between media with refractive indices n_1 and n_2 (Fig. 1.11). Let the

distance $AO = OB = l$. Assume that the x-axis runs along the interface, and the origin of coordinates is at the point O, where the ray strikes the interface. Draw a broken line ACB (point C must lie on the interface between the media). According to Fermat's principle, the time required to traverse ACB will be greater than the time required to traverse the actual path AOB (for which $\sin\alpha/\sin\beta = n_2/n_1$) at any value of $x = OC$. Make sure this is true considering, for the sake of simplicity, sufficiently small values of x.

Using the law of sines for the triangles AOC and BOC, we have

$$AC=\sqrt{l^2+x^2+2lx\sin\alpha}=l\sqrt{1+(\eta^2+2\eta\sin\alpha)}, \tag{1.16}$$

$$CB=l\sqrt{1+(\eta^2-2\eta\sin\beta)} \quad (\eta=x/l).$$

Recall the approximate relation $\sqrt{1+\gamma} = 1 + \gamma/2$ which holds true for $\gamma \ll 1$. Since we are assuming that $x \ll l$ and, consequently, $\eta \ll 1$, we can use the above approximate equation and rewrite (1.16) as

$$AC = l\,(1 + \eta\sin\alpha + \eta^2/2);$$
$$CB = l\,(1 - \eta\sin\beta + \eta^2/2). \tag{1.17}$$

The time T required for the light to traverse AOB is $T = l\,(n_1 + n_2)/c$. Designate the time which it would take the light to traverse the path ACB as T_x, thus $T_x = (AC\cdot n_1 + CB\cdot n_2)/c$.

Substituting (1.17) for AC and CB and using (1.11) (remembering that $\eta = x/l$), we have

$$T_x = \frac{l}{c}(n_1+n_2) + \frac{1}{2}\frac{l}{c}\left(\frac{x}{l}\right)^2 (n_1+n_2) = T + Dx^2.$$

Obviously, $T_x > T$ whatever the sign of x, which is what we set out to prove.

Now let us use Fermat's principle to solve the following problem. *There is a coin at the bottom of a reservoir which has a depth H. We view it from above along a vertical line.*

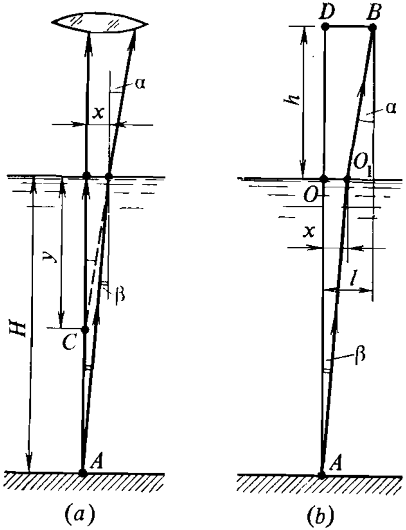

Fig. 1.12.

What is the apparent distance between the water surface and the coin? The refractive index of water n is given.

Figure 1.12*a* shows a greatly magnified crystalline lens of the observer. Two light rays from the coin enter it. One follows a strictly vertical path (it is not refracted), and the other enters it at a very small angle to the vertical (this ray is refracted at the interface between the water and air). The observer sees the coin where the extensions of the diverging rays arriving at the eye converge. The figure shows that this happens at the point C. So, the distance from the water surface to the coin is OC and we designate it as y.

To find the value of y, we have to know the relationship between the angles α and β, which follows from the law of refraction $\sin\alpha/\sin\beta = n$. Since in this instance the angles α and β are *very small*, we can safely use the approximate relations

$$\sin\alpha = \tan\alpha = \alpha, \quad \sin\beta = \tan\beta = \beta. \qquad (1.18)$$

(Note that in (1.18) the angles must be measured in radians and not degrees.) Thus, in the problem under consideration, the law of refraction assumes a particularly simple form:

$$\frac{\alpha}{\beta} = n. \qquad (1.19)$$

It follows from basic geometry (see Fig. 1.12) that $H\beta = x$ and $y\alpha = x$; so $H\beta = y\alpha$. With regard to (1.19), we get

$$y = \frac{H}{n}. \qquad (1.20)$$

Our problem turned out quite simple provided we are familiar with the law of refraction. Now let us assume that we had no knowledge of the law of refraction. Fermat's principle would enable us to deduce (1.19) and through this resolve the problem.

The light ray travels from A to B; assume that $OD = h$, and $DB = l$ (see Fig. 1.12*b*). Designate the point at which the ray is refracted as O_1; $OO_1 = x$. We must determine the value of x for which the time required to traverse the path AO_1B is the least. The time T of transit over this path is described by the equation

$$T = \frac{n}{c}\frac{H}{\cos\beta} + \frac{1}{c}\frac{h}{\cos\alpha}, \qquad (1.21)$$

where c is the velocity of light in vacuum (we hold that the velocity of light in air is the same). Using (1.18), we get

$$\cos\beta = 1 - 2\sin^2\frac{\beta}{2} = 1 - \frac{1}{2}\beta^2;\quad \cos\alpha = 1 - \frac{1}{2}\alpha^2. \qquad (1.22)$$

Since $\xi \ll 1$, the following approximate relation holds true:

$$\frac{1}{1-\xi} = 1 + \xi. \qquad (1.23)$$

Making use of (1.22) and (1.23), we can write (1.21) as

$$T = \frac{nH}{c}\left(1 + \frac{\beta^2}{2}\right) + \frac{h}{c}\left(1 + \frac{\alpha^2}{2}\right).$$

Since

$$\alpha = \frac{l-x}{h} \text{ and } \beta = \frac{x}{H}, \qquad (1.24)$$

we have

$$T = \frac{nH}{c}\left(1 + \frac{x^2}{2H^2}\right) + \frac{h}{c}\left[1 + \frac{(l-x)^2}{2h^2}\right].$$

We must determine the value of x for which T is the least. In other words, we must find the value of x for which the following function reaches its minimum:

$$y(x) = n\frac{x^2}{H} + \frac{(l-x)^2}{h} = \frac{nh+H}{hH}x^2 - 2\frac{l}{h}x + \frac{l^2}{h}.$$

It is known that the x-coordinate of the vertex of the parabola $y = ax^2 + bx + c$ is $b/2a$. Consequently, the value of x we are looking for equals

$$x = \frac{lH}{nh+H}. \qquad (1.25)$$

Substituting (1.25) into (1.24), we have $\alpha = \dfrac{ln}{nh+H}$, $\beta = \dfrac{l}{nh+H}$, whence $\alpha/\beta = n$.

Total Internal Reflection of Light. Critical Angle of Reflection. Up to now, while examining the refraction of light at an interface, we have virtually disregarded the reflection of light from the interface which occurs *simultaneously* with refraction. Strictly speaking, the two phenomena (refraction and reflection of light) should be considered together. This was proved in a most convincing way by the outstanding French scientist Augustin Jean Fresnel (1788-1827) who obtained the relationships for the intensity of the refracted and reflected beams of light with regard to the incident beam's intensity, the magnitude of the angle of incidence, and the polarization of the light. These relationships are known today as *Fresnel's formulae*. They have preserved their original form in modern optics.

Fresnel's formulae go beyond the confines of this book because we would need to use the electromagnetic theory of light to interpret them. Besides, polarization of light needs to be discussed separately. That is why we shall limit ourselves to a few general remarks concerning the interrelations between the intensities of the refracted and reflected beams of light, and examine the case when light passes from a medium with a higher refractive index to a medium with a lower one (in other words, from a dense to a less dense medium). This case is of special interest to us as it illustrates the phenomenon of *total internal reflection*.

Figure 1.13 shows four cases corresponding to different magnitudes of the angle of incidence α of a light beam. Light falls on an interface be-