All a matter of balance — or a problem with dominoes.

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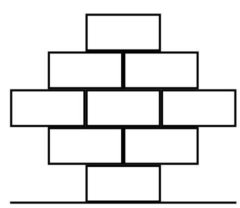
1 Introduction — the problem

The problem[1] is this:

> *"when we stack dominoes with only one touching the table, what is the greatest horizontal distance we can cover keeping in balance?"*

Taking as many dominoes, or other small blocks, as we like, we want to cover the greatest horizontal distance. One solution, after a little thought, will convince you that there is *no limit* to the distance that can be covered. By constructing a cantilever structure, such as that shown in the Figure below we can cover any horizontal distance we choose.



Clearly this problem isn't very *interesting*, what we are really interested in solving is the problem of the maximum distance we can cover using a *leaning stack* of dominoes where:

1. the stack must balance,
2. with only one domino per level.

So how far horizontally can we get a tower of dominoes, one on top of the other, to lean? There are to be no counter balancing or cantilever type structures here!

Taking large dominoes we could obviously cover a large distance, so we think of the "distance covered" in units relative to the width of a single domino. Can we cover a distance twice the width? Three times? Accordingly we assume the width is 2 and the block has mass 1. See Figure 1. The height of the dominoes plays no role.

[1] I would like to thank Dr. C. Good of the University of Birmingham for posing this problem, see also [1].

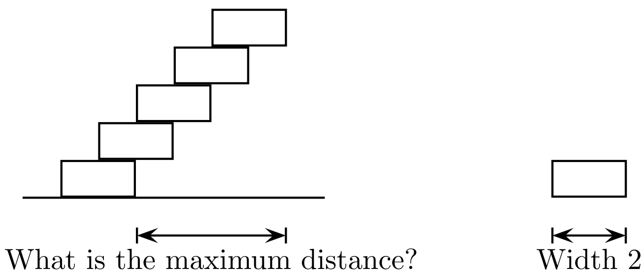
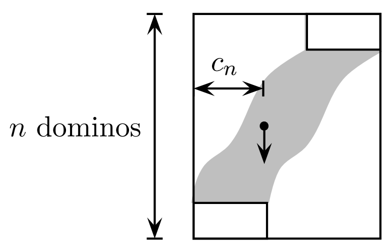


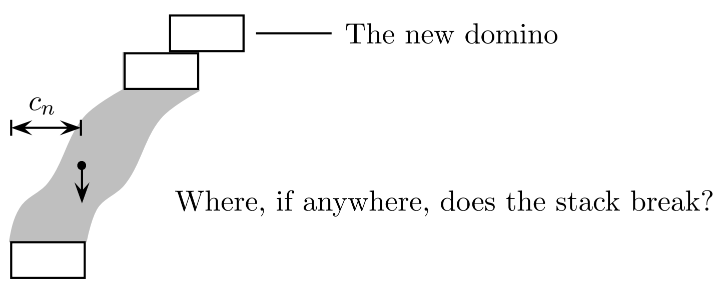
Figure 1: A leaning stack.

2 Describing the problem

Now we have described the problem informally we should think of how to describe it mathematically. Let us assume then that we have a leaning stack of n dominoes. Each domino has width 2 and mass 1. We define c_n to be the horizontal distance from the left most domino to the centre of mass.



We want to add another domino to the stack without destroying it. But this poses a problem. If we add a domino to the top the stack could break at *any point.* So our analysis would need to check that at no point does the stack topple over.



Now, there is a simpler way. We could proceed by induction placing the stack *on top* of the domino making sure we do this in such a way that the centre of mass of the existing stack is above the new domino. See Figure 2. So our strategy is this: at each stage we place the existing (balancing) stack on top of a new domino a distance δ_n from the left of the domino. There will clearly be no toppling if

$$\delta_n + c_n < 2 \quad \text{for all } n. \tag{1}$$

That is to say if we don't displace the top stack so far that the displacement plus the distance of the centre of mass from the left pushes the centre of mass over the edge of the bottom

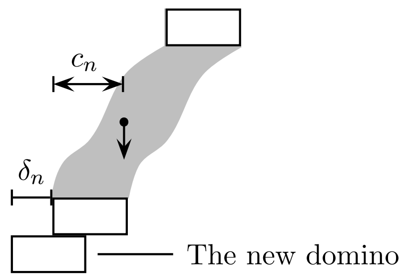


Figure 2: Adding a domino at the bottom of the stack

domino – which has width 2. The new centre of mass of the whole stack of $n+1$ dominoes will be c_{n+1} from the left of the bottom domino where

$$c_{n+1} = \frac{(\delta_n + c_n)n + 1}{n+1}$$

with

$$c_1 = 1 \quad \text{(one domino).}$$

2.1 A reality check

What happens if we build a vertical tower with no displacement at all – that is to say $\delta_n = 0$ for all n. Then from our relation above

$$c_{n+1} = \frac{c_n n + 1}{n+1}$$

Given that $c_1 = 1$ we see, by induction, that $c_n = 1$ for all n also. So for a vertical stack, the centre of mass is always in the middle – as expected.

2.2 One non-trivial strategy

Given our no-toppling condition (1) we could *choose* to take δ_n to be

$$\delta_n := \frac{2 - c_n}{2} \tag{2}$$

which is half the distance needed each time to the topping point. Taking this scheme, what is δ_n? and more importantly the total horizontal distance which equals $\sum_n \delta_n$?

Firstly let's calculate the c_n's explicitly under this scheme.

$$c_{n+1} \quad = \quad \frac{(\delta_n + c_n)n + 1}{n+1} \quad = \quad \frac{(2 + c_n)n + 2}{2(n+1)} \quad = \quad \frac{nc_n}{2(n+1)} \quad + \quad 1.$$

Now we set $b_n := 2 - c_n$ and see what happens to the the b_n's.

$$b_{n+1} = 2 - c_{n+1} = 1 - \frac{nc_n}{2(n+1)} = 1 - \frac{n(2-b_n)}{2(n+1)} = \left(1 - \frac{n}{n+1}\right) + \frac{b_n}{2}\frac{n}{n+1} \geq \frac{1}{n+1}.$$

So by induction

$$b_n \geq \frac{1}{n}.$$

But,

$$\delta_n := \frac{2-c_n}{2} = \frac{b_n}{2} \geq \frac{1}{2n}.$$

So we see that the total displacement is $\sum_n \delta_n$ and if we take N dominoes in the stack this exceeds

$$\sum_{n=1}^{N} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{N} \frac{1}{n}. \tag{3}$$

So that the question becomes, what is the value of

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N} = \sum_{n=1}^{N} \frac{1}{n}$$

for large N? If $N \to \infty$, this is a particularly famous series – the *harmonic series* – which diverges. That is to say it is possible to make the sum (3) as large as one would wish. In terms of the domino problem: we can chose displacements δ_n so that (i) the stack does not topple over, and (ii) we can produce an arbitrarily large horizontal displacement. Bizarre indeed!

2.3 The medieval proof of the divergence of the harmonic series

The following medieval proof that the harmonic series diverges was discovered and published by Orseme around 1350 and relies on grouping the terms in the series as follows

$$\begin{aligned} &1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \cdots \\ &\quad = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} + \tfrac{1}{12} + \tfrac{1}{13} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{16}\right) + \cdots \\ &\qquad \geq 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \cdots \end{aligned}$$

It follows that the series can be made arbitrarily large. In fact this series diverges quite slowly. Calculus can be used to provide an explicit estimate on the rate of growth by comparing the graph of a function with the terms of the series. By integrating the function we can compare the sum of the series with the integral of the function and draw conclusions from this.

In this case we compare terms in the series with the area under the graph of the function $1/(1+x)$. In particular Figure 3 shows that

$$\sum_{k=1}^{n} \frac{1}{k} > \int_0^n \frac{1}{1+x}\mathrm{d}x.$$

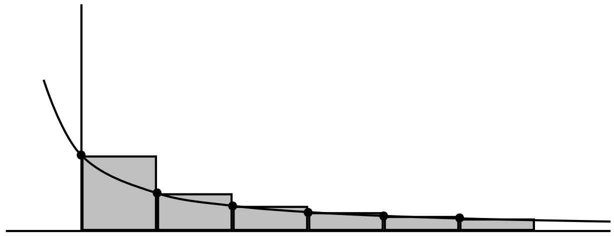


Figure 3: Comparing the series $1/n$ with the function $1/(1+x)$.

Of course the integral on the right can be solved giving

$$\sum_{k=1}^{n} \frac{1}{k} > \ln(1+n).$$

Now, the function $\ln(1+n)$ is unbounded so that we can make $\sum_{k=1}^{n} \frac{1}{k}$ as large as we please. A similar argument comparing the series to the function $1/x$ shows that

$$1 + \ln(n) > \sum_{k=1}^{n} \frac{1}{k} > \ln(1+n) \tag{4}$$

so that we can estimate how fast the series diverges.

3 Building the stack of dominos

It is perhaps inevitable that having realized that there is no limit to the potential horizontal lean of the stack of dominoes we might like to see what it *looks like*. Can we really construct such a stack in practice and if so, how would we do this?

Remember that to construct the stack using our scheme[2] we take

$$\delta_n := \frac{2 - c_n}{2}$$

where c_n is calculated from the recurrence relation

$$c_{n+1} = \frac{nc_n}{2(n+1)} + 1 \text{ with } c_1 := 1.$$

What we really want is an explicit recurrence relation for δ_n and we can calculate this as follows:

$$2\delta_{n+1} = 2 - c_{n+1} = 2 - \frac{nc_n}{2(n+1)} - 1 = \frac{2n + 2 - 2n(1-\delta_n)}{2(n+1)} = \frac{1 + n\delta_n}{n+1}$$

Thus

$$\delta_{n+1} = \frac{1 + n\delta_n}{2(n+1)} \text{ with } \delta_1 := \frac{1}{2}.$$

[2] Recall that our scheme for choosing δ_n was only one of many possibilities – some will give finite horizontal distances (take $\delta_n := 2^{-n-1}$) and others, such as ours, have no limit.

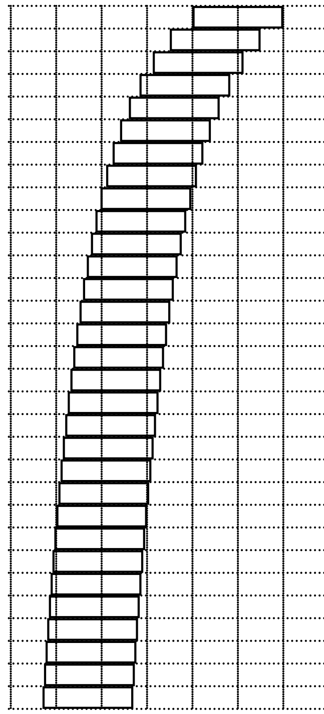


Figure 4: The diverging stack of dominoes.

Unfortunately this recurrence relation does not seem to have a simple closed form solution[3]. Thus we calculate successive values of δ_n numerically which are shown in Table 1. Notice also how close the value of c_n gets to the critical value of 2 – the value above which the stack will over balance the bottom domino.

If we use this recurrence relation we construct the leaning tower shown in Figure 4. Notice that in this picture the blocks have width 2 and height 1/2.

3.1 Toppling the stack!

Given that c_n approaches the critical value of 2 so closely we examine what *angle* through which we would need to rotate the whole stack about the bottom corner to topple the stack.

The centre of mass of an object is fixed within the object regardless of how we rotate the object. For our stack of dominoes this is c_n from the left, and so $2 - c_n$ from the right of the bottom block, as illustrated in Figure 5. The blocks have width 2 but to this point we have not mentioned their height. We take the height of a block to be h so that the centre of mass

[3]If we take the initial condition $\delta_1 = 1$ instead then a simple induction argument shows that $\delta_n = 1/n$ for all n. Notice that the recurrence relation $h_{n+1} = nh_n/(n+1)$, with $h_1 = 1$ has solution $h_n = 1/n$. The relation

$$\delta_{n+1} = \frac{1 + n\delta_n}{2(n+1)}$$

is in some vague sense an average between the explicit solution $\delta_n = 1/n$ and a relation that generates this. Furthermore, if we take $\delta_1 = 1$ then $c_n = 2$ for all n by induction also. See Section 4 below.

n	c_n	δ_n	θ_n
1	1.000	0.500	75.964
2	1.250	0.375	56.310
3	1.417	0.292	37.875
4	1.531	0.234	25.115
5	1.613	0.194	17.223
6	1.672	0.164	12.339
7	1.717	0.142	9.201
8	1.751	0.125	7.097
9	1.778	0.111	5.630
10	1.800	0.100	4.569
11	1.818	0.091	3.781
12	1.833	0.083	3.179
13	1.846	0.077	2.710
14	1.857	0.071	2.337
15	1.867	0.067	2.036
16	1.875	0.062	1.790
17	1.882	0.059	1.586
18	1.889	0.056	1.414
19	1.895	0.053	1.270
20	1.900	0.050	1.146
21	1.905	0.048	1.039
22	1.909	0.045	0.947
23	1.913	0.043	0.866
24	1.917	0.042	0.796
25	1.920	0.040	0.733

Table 1: The diverging stack of dominoes.

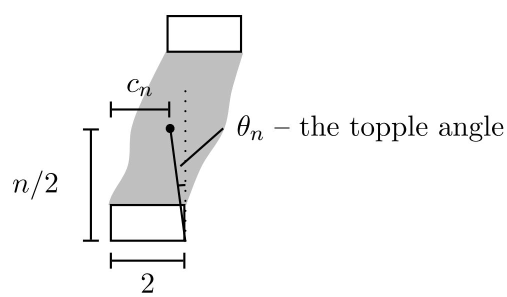


Figure 5: Toppling the stack

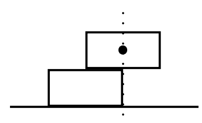
is $hn/2$ from the bottom of the stack. Thus we see that the angle of rotation θ_n needed to topple the block is

$$\theta_n = \arctan\left(\frac{4 - 2c_n}{hn}\right).$$

When $h = 1/2$ (so the width of the block is four times the height) values for θ_n, in degrees, are also given in Table 1.

4 ***"Weebles wobble but they don't fall down"***

Our original strategy (2) for choosing δ_n is on reflection rather conservative. We only push the top stack *half way* to the balancing point. What happens if we push it *all the way?* By this we mean we move the centre of mass so that it is right on the edge of the bottom domino. For two dominoes this means $\delta_1 = 1$ and we have the following configuration.



Remember that a domino will only topple if there is a positive rotational force generated by the centre of mass being *over the edge*. An arbitrary small displacement to the right or rotation clockwise will topple the stack in this situation but for the moment, that it balances in theory is sufficient.

To continue this we would need, this time, to choose

$$\delta_n := 2 - c_n. \tag{5}$$

If we do this and derive our series for δ_n we find that

$$c_{n+1} = \frac{(\delta_n + c_n)n + 1}{n+1} = \frac{(2 - c_n + c_n)n + 1}{n+1} = \frac{2n+1}{n+1}$$

Using this in (5) gives

$$\delta_{n+1} = 2 - c_{n+1} = \frac{1}{n+1} \text{ with } \delta_1 = 1,$$

i.e. $\delta_n = 1/n$ for all n. So that if we choose to place the top stack right on the balancing point, the resulting displacements are *precisely the terms in the harmonic series.*

This also illustrates the difference between stability and instability. Our original scheme (2) gave rise to a stable stack – there exists a small but positive displacement through which you can rotate the stack without toppling it. If we choose the scheme (5) *any* displacement to the right, however small, will topple the stack. See Figure 6.

Not only can we build a stack that leans arbitrarily far but we can build a stable stack. As [2] comments

> To prove this result "physically", a fellow graduate student and I stacked bound volumes of *The Physical Review* one evening, until an astonishingly large offset was obtained, and left them to be discovered the next morning by a startled physics librarian.

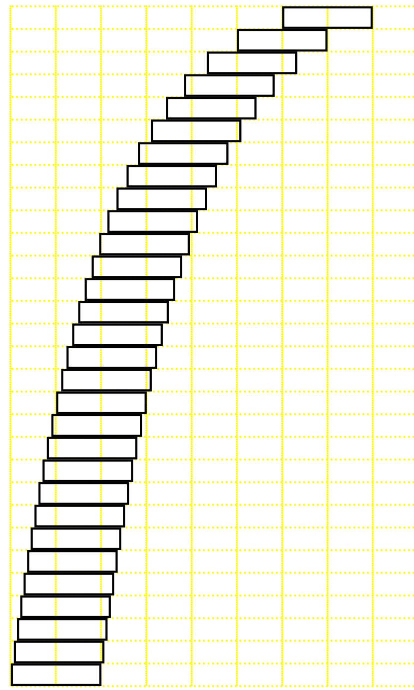


Figure 6: The stack of dominoes.

5 Spiraling out of control

When the problem was presented in Section 1 we assumed, tacitly at least, that the greatest horizontal displacement would be obtained if all the displacements occurred in one direction. Furthermore it was assumed that the stack occupied a vertical plane. Having solved that problem we might wonder whether we could build a self supporting "spiral stair case". As before there is one domino per level but where we now travel in different directions in the horizontal plane.

The answer, perhaps even more surprisingly is that we can do *quite a lot more* than just that. In particular if you specify an ordered sequence of points $(x_1, y_1), ..., (x_N, y_N)$ in the horizontal plane it is possible to build a self supporting stack of dominoes, with only one per level (as before), which passes above each of these points. If you like, the leaning stack is the very specialized case where these was just one point – $(N, 0)$. That is we want to build a stack which passes above the point $(N, 0)$ for any N.

The key to understanding this is to return to the stack shown in Figure 6, which was generated using (5). We imagine it to be two separate stacks one on top of the other. Both balance and the center of mass of the upper stack is above the top domino of lower stack. If we rotate the top stack in a horizontal plane about a line with runs through the centre of mass of the top stack two things remain true, (i) the position of the centre of mass of the top stack doesn't move, and (ii) it hence remains above the top domino of the lower stack. See Figure 7.

However this argument is not quite sufficient – none of the dominoes are glued and so we may not simply lump the dominoes comprising the top part and treat them as a single entity. Why, for example, doesn't the bottom domino fall out?

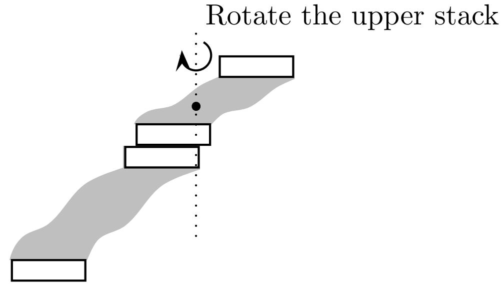
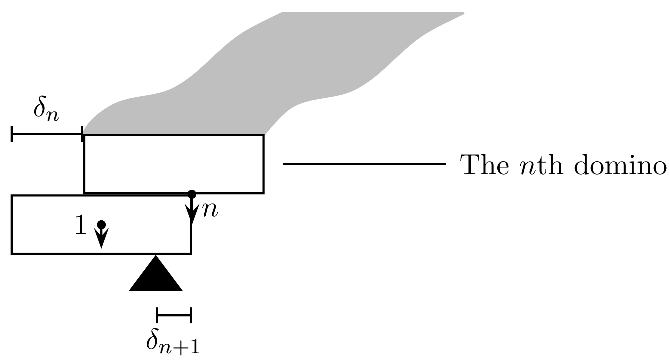


Figure 7: Rotating part of the stack in the horizontal plane

We recall the scheme of Section 4, where we chose the δ_n's so that the upper stack was always placed with the centre of mass right on the edge of the domino below it. In effect we found where we could balance the upper stack. Our induction argument can be changed to give us more: not only does the stack fail to topple over to the right but it balances on this point.

Let us examine the situation when we have $n+1$ dominoes. The $n+1$th of these is placed on a balancing point $\delta_{n+1} = \frac{1}{n+1}$ from it's right hand end. Above it are n dominoes with a centre of mass exactly at the end. Recall we chose δ_n so that $c_n = 2$ for all n. The forces on the $n+1$th domino are summarized below.



Resolving about the lower balancing point we see that the forces about the triangular pivot (which will be the edge of the $n+2$th domino) are

$$1 \times (1 - \delta_{n+1}) - n \times \delta_{n+1} = 1 - \frac{1}{n+1} - \frac{n}{n+1} = 0.$$

That is to say the forces balance and the $n+1$th domino is held in balance on this pivot point precisely by the mass of the dominoes above it.

This construction is unstable and the resulting stack, such that in Figure 8, would fall over with any displacement of the top domino. Of course, we could perform such a rotation at any – indeed *every* – level if we so chose. Furthermore we may start with an initial configuration covering an arbitrary horizontal distance in one direction. Thus we can rotate parts of the stack and build a spiral staircase, or any other similar shape we choose to build.

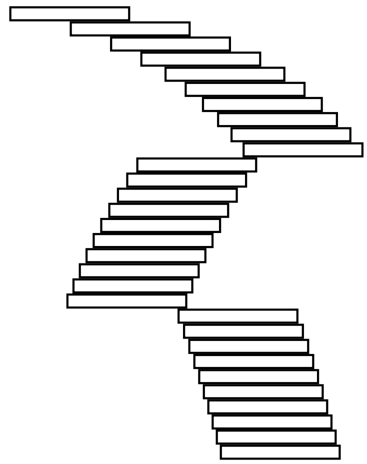


Figure 8: Even this balances

6 The leaning pencil and reaching the stars

The width of the dominoes was set arbitrarily at 2. This need not, of course, be the case. We could make the width as big or as small as we like, alter the argument and get our stack to balance.

If the stack balances there is no reason why we can't glue it together and just balance it on the base. Further we might be able to, if we took very small blocks, to sand it down to give a smooth finish. Then we'd have a bent pencil like wooden shape that leans as far as we like across the table but balances on it's base. Ok, it will be rather tall, and won't take much to topple it over, *but it will balance!*

Now, this all seems a little too weird to believe so, suppose we wish to span a distance of 3 metres with CD cases, which actually balance rather well. A CD case is about $14\text{cm}\times 12\text{cm}\times 1\text{cm}$. So we want

$$\sum_{n=1}^{N} \frac{1}{n} = 3\text{m}$$

where the left hand side is in units of half-a-CD-case, which is about 7cm.

$$\sum_{n=1}^{N} \frac{1}{n} = 300/7 = 42.857...$$

By the approximation (4) this gives

$$N = e^{42.857...} = 4.1 \times 10^{18}.$$

Now the thickness of a CD case was taken to be 1cm, so the height of the corresponding stack of CD cases would be

$$4.1 \times 10^{16}\text{m} = 4.3$$

light years (one light year is 9.46×10^{15} m) which just happens to be the distance between here and Alpha Centuri. If you tried to build it the gravitational attraction between the CD cases would probably be much more significant than the Earth's gravity on the CD cases. Making the problem of "balance" more tricky!

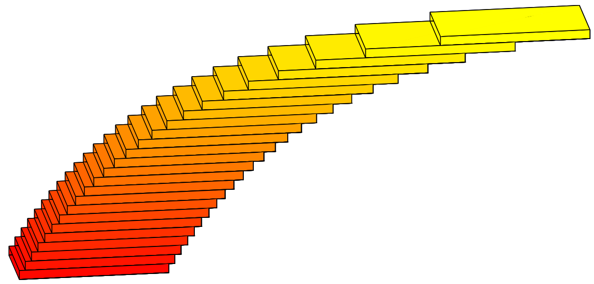


Figure 9: An unlikely looking, but never the less balancing stack!

References

[1] Johnson, P. B., *Leaning tower of Lire,* American Journal of Physics, 22, (1) pp. 240, April, 1955.

[2] Eisner, L., *Leaning tower of The Physical Reviews,* American Journal of Physics, 27, (2) pp. 121–122, February, 1959.

[3] Gardner, M. *Martin Gardner's Sixth Book of Mathematical Games from Scientific American.* New York: Scribner's, p. 167, 1971.