Search for Simplicity: Mountains, waterwaves, and leaky ceilings

We would like to find a reason why the highest elevations of mountains on Earth are of the order of 10 km. It must be connected with the rigidity of rock. We assume that the volcanic and tectonic activity of the Earth produces elevations and depressions of the Earth's surface. What limits the height of the elevations? We simplify a mountain by a block of silicon oxide resting on a plane surface of the same material (see Fig. 1). The mountain will be too high to be supported by the base when the block is so large that the base matter starts to flow; the mountain will have reached its maximum height when plastic deformation sets in.

Let H be the height of our block at which it begins to sink. The H will be roughly the maximum height which a mountain can reach. This height is reached when the energy gained by letting the mountain sink is equal to the energy necessary to engender plastic flow. The amount of matter undergoing plastic deformation is about equal to the amount of mountain matter which sinks into the ground. The sinking of the mountain by the amount $\delta \ll H$ is equivalent to moving a layer of thickness δ from the top into the ground, displacing a comparable volume by plastic flow. Hence, roughly speaking, the amount of gravitational energy gained by lowering matter from the height H to the ground must be equal to the energy for plastic deformation of the same amount of matter. The calculation can be done for each molecule separately. Let us call ϵ_p the energy per molecule necessary to induce plastic flow. Then we get the relation

$$AmHg = \epsilon_p \,, \tag{1}$$

where $A = 60$ is the molecular weight of SiO_2, m is the mass of a proton, and g is the gravitational acceleration on Earth.

How can we get an estimate of ϵ_p? Plastic flow is a rearrangement of the molecules. When it occurs the molecules must pass through spatial arrangements that would not occur in the solid phase but rather in the liquid state. We get an idea of the energy necessary to get to these states by melting the substance and then supercooling the liquid back to the original temperature. Assuming that the heat capacities of the solid and the supercooled liquid state are the same, the energy necessary for this process is the melting heat ϵ_M at melting temperature. For SiO_2 $\epsilon_M = 0.148$ eV per molecule. We therefore conclude that $\epsilon_p \approx 0.15$ eV. We then obtain $H \approx 14$ km. This result is surprisingly close to the actual value of 10 km.

This estimate is based upon a competition between two effects: the gravitational force and the rigidity of the material. The energy ϵ_p measuring the rigidity is connected with atomic energies. It is a fraction $\xi\epsilon_B$ of the binding energy $\epsilon_B = 6.5$ eV (the energy needed to extract a molecule from solid SO_2). As expected, the coefficient ξ is rather small: $\xi = 0.023$. The binding energy $\epsilon_B = \eta$ Ry is of the order, but smaller, than the Rydberg energy $\text{Ry} = me^4/2\hbar^2$, with $\eta = 0.48$. Expressed in fundamental constants, we get

$$H = \xi\eta \,\text{Ry}/(Amg) \,. \tag{2}$$

Let us now look at two other seemingly very different natural phenomena: the size of drops on a leaky ceiling, and water waves on the surface of a lake. What is the size of the drops forming on a leaky ceiling when they fall down? The leaking water forms a thin film on the surface which is unstable. A slight accumulation at one point starts growing downwards by water flowing into it from all sides, since this reduces the gravitational energy (see Fig. 2). When will the drop come off?—when the gravity force becomes larger than the surface tension that keeps the drop on the ceiling. The surface tension S is an energy per unit area or a force per unit length. Let us approximate the drop as a hemisphere of a radius R pointing downwards. Then the force F holding it up is the surface tension along the periphery where the drop merges with the film on the surface: $F = 2\pi RS$. When this force becomes equal to the gravity force $(2\pi/3)R^3\rho g$, the drop will fall. Here, ρ is the density of water. We then get for the radius of the drop ($S = 73$ dyn/cm in water):

$$R \approx (3S/g\rho)^{1/2} = 0.47 \text{ cm} \,. \tag{3}$$

The result is not exact since the form of the drop when attached to the ceiling deviates from a hemisphere, in particular shortly before separation. However, it does give a size of the drops not far from the one we do observe all too frequently.

Let us now turn to the water waves. When a light breeze starts blowing over a quiet surface of a lake, the wavelength λ of the initial waves is of the order of a few centimeters. "Willows whiten, aspens quiver, little breezes dusk and shiver" as the poet Tennyson says. We will not enter into the physics of wave production; suffice it to say that the wind transfers its energy first to those waves whose propagation velocity v is lowest. The expression for v is

$$v = (g\lambdabar + S/\lambdabar\rho)^{1/2} \,,$$

where S is the surface tension, ρ the density of water, and $\lambdabar = \lambda/2\pi$. The first term comes from the gravity and the second term from the surface tension. It is evident that the longer $\lambdabar$, the stronger gravity acts as a restoring force, and the smaller $\lambdabar$ the more the curvature of the surface causes a restoring force. The minimum of v occurs at

$$\lambdabar_m = (S/g\rho)^{1/2} = 0.28 \text{ cm} \,. \tag{4}$$

The corresponding minimum value v_m is 23 cm/s. A wind with less than this speed would be unable to produce waves.

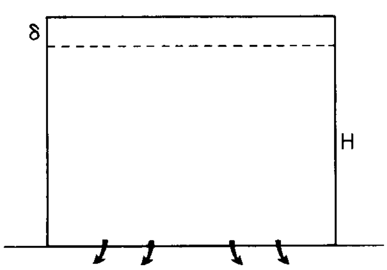

Fig. 1. The sinking of the mountain by the amount δ corresponds to the displacement of a layer of thickness δ from the top into the ground, and to a plastic flow of a comparable volume in the ground.

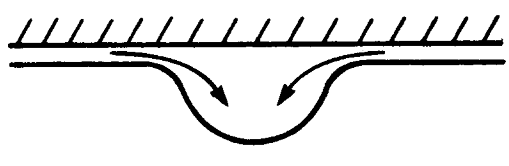

Fig. 2. The forming of a water drop from a thin water film below a surface.

This is why lakes are so much like mirrors, even in the presence of weak winds. The first waves appear when the breeze surpasses v_m, and should have a wavelength of $2\pi\lambdabar_m$. Actually the minimum velocity v_m is somewhat smaller and the corresponding wavelength λ_m is somewhat larger, because we should have done our estimate by using the group velocity instead of the phase velocity of the waves, but the order of magnitude is the same. We would have found $v_m = 18$ cm/s, and $\lambda_m = 4.4$ cm. Note that $\lambdabar_m$ is the same length as the radius of the falling drop as given by (3) apart from a factor $\sqrt{3}$ for the phase velocity calculation or a factor 0.68 for the group velocity calculation.

In order to compare these results with the mountain height, we express S and ρ in terms of molecular properties. The binding of a molecule at the surface is $(1-\xi')$ times the binding energy ϵ_B in the interior. In the January 1985 installment,[1] we found that ξ' is about $\frac{1}{6}$. In water it is somewhat smaller: $\xi = 0.093$. The surface tension expressed as energy per unit surface is $S = \xi'\gamma' \,\mathrm{Ry}\cdot d^{-2}$, where $\epsilon_B = \gamma'$ Ry and d is the distance between molecules so that d^{-2} is the number per cm^2 of surface. For water, $\gamma' = 0.037$, and the density is $\rho = A'm/d^3$; where A' is the molecular weight of water $A' = 18$. We then get

$$\lambdabar_m^2 = \frac{R^2}{3} = \frac{\xi'\eta' \,\mathrm{Ry}}{gA'm} d = C\cdot d\cdot H, \quad C = \frac{\eta'\eta' A}{\eta\eta A'},$$

where the second equality comes from a comparison with (2). The factor C is not too far from unity: $C = 1.8$. In other words, λ_m and the size of falling drops R are roughly the geometric mean between the maximum height of the mountains and the intermolecular distance d.

How about the maximum height of mountains on other planets? For this aim we must express the earthbound quantity g by means of Newton's constant G: $g = GM/R_P^2$, where M and R_P are the mass and radius of the Earth. We put $M = (4\pi/3)R_P^3\rho_P$, where $\rho_P = 5.5\ \mathrm{g/cm^3}$ is the mean density of the planet. We then get the value $g = (4\pi/3)G\rho_P R_P$, and from (2):

$$H = R_0^2/R_P, \quad R_0^2 = (3/4\pi)\xi\eta \,\mathrm{Ry}/(AmG\rho_P). \quad (5)$$

With the values of ξ, η, and ρ valid for the Earth, the length R_0 is 300 km. Equation (5) tells us that the maximum elevations on different planets are inversely proportional to the radius if the material properties are nearly the same, as they are more or less for Mars and our moon. Thus the mountains and valleys on Mars ought to be about twice as high or deep than on Earth. This is indeed the case. The elevations on the moon ought to be six times larger; actually they are much lower. This is because, in contrast to the Earth and Mars, the moon has had no tectonic activity for a long time, so that its surface has been flattened by meteoric erosion and other causes.

According to (5), a celestial body of a radius $R \lesssim R_0$ would sustain mountains comparable to its radius. Such a body could be nonspherical if no liquefaction took place after it assumed such a shape. The gravity would be no longer able to produce an approximate sphere by plastic deformation. The critical radius is $R \sim 300$ km if the material composition is similar to that of the Earth. Indeed all known nonspherical celestial bodies, such as the moons of Mars and some moons of Saturn, are smaller than R_0.

It is instructive to express H or λ_m not in an anthropomorphic unit such as kilometer but in terms of the only "dignified" unit among atoms: the Bohr-radius $a = \hbar^2/(me^2) = 0.53 \times 10^{-8}$ cm. Also, let us get rid of all nonatomic quantities in expression (5) such as the density ρ_P and the radius R_P of the planet, and express them in terms of N_P, the number of nucleons (protons and neutrons) in the planet. Fortunately, the main constituents SiO^2 and Fe have a very similar molecular weight, say, $A \sim 60$. We introduce a distance d between them, which is defined by $(N_P/A)d^3 = (4\pi/3)R_P^3$, and express it as a multiple of the Bohr radius: $d = fa$, where f turns out to be 4.9 for the Earth. We then get altogether for the maximum height of mountains in terms of the Bohr radius

$$\frac{H}{a} = 0.19\xi\eta f^2 \frac{\alpha}{\alpha_g} \frac{1}{A^{5/3}N_P^{1/3}} = 2.6 \times 10^{14}. \quad (6)$$

Here α/α_G stands for e^2/GM^2, whereby $\alpha = e^2/\hbar c = (137)^{-1}$ and $\alpha_G = GM^2/\hbar c = 6.1 \times 10^{-39}$. The former is the electromagnetic fine structure constant, the latter is its analog for gravity. The appearance of the ratio of these constants is an indication that H is the result of a competition between the atomic forces which are electric effect governed by quantum mechanics, and gravity effects. The smallness of α_G is compensated by the large number $N_P^{1/3}$. The square root of (6) gives the wavelength λ_m produced by a breeze on a lake, and the radius of a falling drop, in terms of the Bohr-radius. Adjusting the constants ξ, γ, f, and A to water, we get $\lambdabar_m/a \approx 10^7$. The greatness of mountains, the finger sized drop, the shiver of a lake, and the smallness of an atom are all related by simple laws of nature.

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[1] Am. J. Phys. **53**, 19 (1985).